Orthogonal Polynomials for Potentials of two Variables with External Sources

M.C. BERGERE

Service de Physique Théorique, CEA-Saclay
F-91191 Gif sur Yvette cedex, France
email: bergere@spht.saclay.cea.fr

February 1, 2008

Abstract

This publication is an exercise which extends to two variables the Christoffel's construction of orthogonal polynomials for potentials of one variable with external sources. We generalize the construction to biorthogonal polynomials. We also introduce generalized Schur polynomials as a set of orthogonal, symmetric, non homogeneous polynomials of several variables, attached to Young tableaux.

1. Introduction

Recent progress in string theory [1] and in quantum chromodynamics [2] have reactualized the study of the spectral statistical properties of random matrices. Although most interesting results are expected from infinitely large matrices, exact results on the expectation values of ratios of random matrix determinants have been obtained for finite matrices, using the technique of orthogonal polynomials.

In this context, considerable results $[9-10-11]$ collected recently in Ref. [3] can be found for potentials $V(x)$ of one variable but relatively few and recent results [4] are known for potentials of two variables $V(x,y)$ or equivalently $V(z,z^*)$ where $z = x + iy$; however, such results could be of some physical interest, for instance, in the BMN program in string theory $[5-6]$.

It is the purpose of this publication to extend to potentials $V(z,z^*)$ with external sources, the known results for potentials of one variable. Later on, as it is already done for potentials of one variable, we expect to generalize this structure to potentials $V(z,z^*)$ with external sources both at the numerator and at the denominator, using the so called Cauchy transform.

This text is organized as follows: the various results are collected in Sect. 1, 2 and 3; the proofs are exposed in Sect. 4, 5, 6 and in the appendix. Finally, a discussion is proposed in Sect. 7 on orthogonal polynomials for potentials of one and two variables.

The construction of orthogonal polynomials for potentials of one variable with external sources is described as follows: given a real potential $V(x)$ which admits an infinite set $\{p_n(x)\}$ of orthogonal monic polynomials (n is the degree in x of the polynomial) satisfying

$$\int dx \; p_m(x) \; p_n(x) \; e^{-V(x)} = h_n \; \delta_{nm} \tag{1}$$

we consider L external sources $(\xi_1, ..., \xi_L)$ and we look for orthogonal monic polynomials $q_n(x; \xi_i)$ such that

$$\int dx \; q_m(x; \xi_i) \; q_n(x; \xi_i) \prod_{i=1}^{L}(x - \xi_i) \; e^{-V(x)} = k_n(\xi_i) \; \delta_{nm} \tag{2}$$

The solution to this problem is known since Christoffel [7] and reads

$$q_n(x; \xi_i) = \frac{1}{\prod\limits_{i=1}^{L}(x - \xi_i)} \begin{vmatrix} p_{n+L}(x) & p_{n+L}(\xi_1) & \cdots & p_{n+L}(\xi_L) \\ \cdots & \cdots & \cdots & \cdots \\ p_{n+1}(x) & p_{n+1}(\xi_1) & \cdots & p_{n+1}(\xi_L) \\ p_n(x) & p_n(\xi_1) & \cdots & p_n(\xi_L) \end{vmatrix} \frac{1}{[n,L](\xi_i)} \tag{3}$$

where the determinant

$$[n, L](\xi_i) = \begin{vmatrix} p_{n+L-1}(\xi_1) & \cdots & p_{n+L-1}(\xi_L) \\ \cdots & \cdots & \cdots \\ p_n(\xi_1) & \cdots & p_n(\xi_L) \end{vmatrix} \tag{4}$$

is needed to make the polynomial $q_n(x; \xi_i)$ monic.

The determinant in (3) is a polynomial in x of degree $n + L$ which vanishes for $x = \xi_i$, $i = 1, ..., L$; therefore, it can be divided by $\prod\limits_{i=1}^{L}(x - \xi_i)$ to give a polynomial in x of degree n. The proof of orthogonality is very simple: we suppose in (2) that $m < n$

$$q_m(x; \xi_i) = p_m(x) + \sum_{r=m-1}^{0} a_r(\xi_i) \; p_r(x) \tag{5}$$

$$q_n(x; \xi_i) \prod_{i=1}^{L}(x - \xi_i) = p_{n+L}(x) + \sum_{r=n+L-1}^{n+1} b_r(\xi_i) \; p_r(x)$$

$$+ (-)^L \frac{[n+1, L](\xi_i)}{[n, L](\xi_i)} \; p_n(x) \tag{6}$$

2

The orthogonality of the polynomials $p_n(x)$ ensures the orthogonality of the polynomials $q_n(x; \xi_i)$.

The pseudonorm $k_n(\xi_i)$ is found to be the ratio of two determinants

$$k_n(\xi_i) = (-)^L \ h_n \ \frac{[n+1, L](\xi_i)}{[n, L](\xi_i)} \tag{7}$$

Now we state a similar result for real potentials of two variables x and y.

We introduce the complex variables $z = x + iy$, $z^* = x - iy$, and we consider a real potential $V(z, z^*)$ which admits an infinite set $\{p_n(z)\}$ of orthogonal monic polynomials

$$\int \int d^2z \ p_m^*(z) \ p_n(z) \ e^{-V(z, z^*)} = h_n \ \delta_{nm} \tag{8}$$

Everywhere in this text, $f^*(z)$ is a short notation for $[f(z)]^*$. If the domain of integration and the potential are rotational invariant ($V(zz^*)$), the set of orthogonal polynomials are sometimes called Ginibre's polynomials [8], namely $p_n(z) = z^n$ (of course, the weights h_n are different in each case); in general, the polynomials are not homogeneous and have complex coefficients.

We now consider L external complex sources $(\xi_1, ..., \xi_L)$ and we look for orthogonal monic polynomials $q_n(z; \xi_i; \xi_i^*)$ such that

$$\int \int d^2z \ q_m^*(z; \xi_i; \xi_i^*) \ q_n(z; \xi_i; \xi_i^*) \ \prod_{i=1}^{L} |z - \xi_i|^2 \ e^{-V(z, z^*)} = k_n(\xi_i; \xi_i^*) \ \delta_{nm} \tag{9}$$

We show in Sect. 5 that the solution is given by the polynomials

$$q_n(z; \xi_i; \xi_i^*) = \begin{vmatrix} Q_{n+L}(z; \xi_i^*) & Q_{n+L}(\xi_1; \xi_i^*) & ... & Q_{n+L}(\xi_L; \xi_i^*) \\ ... & ... & ... & ... \\ Q_{n+1}(z; \xi_i^*) & Q_{n+1}(\xi_1; \xi_i^*) & ... & Q_{n+1}(\xi_L; \xi_i^*) \\ Q_n(z; \xi_i^*) & Q_n(\xi_1; \xi_i^*) & ... & Q_n(\xi_L; \xi_i^*) \end{vmatrix}$$

$$\times \prod_{i=1}^{L} (z - \xi_i)^{-1} \ . \ [<n, L>(\xi_i; \xi_i^*)]^{-1} \tag{10}$$

where the determinant

$$<n, L>(\xi_i; \xi_i^*) = \begin{vmatrix} Q_{n+L-1}(\xi_1; \xi_i^*) & ... & Q_{n+L-1}(\xi_L; \xi_i^*) \\ ... & ... & ... \\ Q_n(\xi_1; \xi_i^*) & ... & Q_n(\xi_L; \xi_i^*) \end{vmatrix} \tag{11}$$

is needed to make the polynomial $q_n(z; \xi_i; \xi_i^*)$ monic.

3

The polynomials $Q_n\left(z;\xi_i^*\right)$ are monic, of degree n in z and are defined from the property

$$\int\int d^2z\ p_m^*\left(z\right)\ Q_n\left(z;\xi_i^*\right)\ \prod_{i=1}^{L}\left(z^*-\xi_i^*\right)\ e^{-V\left(z,z^*\right)}=0\ \ \text{for } m<n \quad (12)$$

and of course the corresponding complex conjugate properties. The proof of the orthogonality of the polynomials $q_n\left(z;\xi_i;\xi_i^*\right)$ is exactly similar to the proof for a potential of one variable but here, of course, it relies upon the existence of the polynomials $Q_n\left(z;\xi_i^*\right)$ satisfying (12). It is the purpose of Sect. 4 to show the existence, to construct and to give the properties of the polynomials $Q_n\left(z;\xi_i^*\right)$. For one external source ξ we show that

$$Q_n\left(z;\xi^*\right)\ =\ \frac{h_n}{p_n^*\left(\xi\right)}\ K_n\left(z,\xi^*\right) \quad (13)$$

$$K_n\left(z,\xi^*\right)\ =\ \sum_{i=0}^{n}\frac{1}{h_i}\ p_i^*\left(\xi\right)\ p_i\left(z\right) \quad (14)$$

In the general case with L external sources, we prove that the monic polynomials

$$Q_n\left(z;\xi_i^*\right)=\frac{h_n}{\left[n,L\right]^*\left(\xi_i\right)}\begin{vmatrix} p_{n+L-1}^*\left(\xi_1\right) & \cdots & p_{n+L-1}^*\left(\xi_L\right) \\ \cdots & \cdots & \cdots \\ p_{n+1}^*\left(\xi_1\right) & \cdots & p_{n+1}^*\left(\xi_L\right) \\ K_n\left(z,\xi_1^*\right) & \cdots & K_n\left(z,\xi_L^*\right) \end{vmatrix} \quad (15)$$

satisfy (12) and are unique. Clearly, all $K_n\left(z,\xi_i^*\right)$ in (15) could be replaced by the corresponding set $K_q\left(z,\xi_i^*\right)$ for any q satisfying $n\le q<n+L$.

We show in Sect. 5 that the polynomials $q_n\left(z;\xi_i;\xi_i^*\right)$ may be written as

$$
\begin{aligned}
&q_n\left(z;\xi_i;\xi_i^*\right)\\
=\ &\begin{vmatrix} p_{n+L}\left(z\right) & p_{n+L}\left(\xi_1\right) & \cdots & p_{n+L}\left(\xi_L\right) \\ K_{n+L-1}\left(z,\xi_1^*\right) & K_{n+L-1}\left(\xi_1,\xi_1^*\right) & \cdots & K_{n+L-1}\left(\xi_L,\xi_1^*\right) \\ \cdots & \cdots & \cdots & \cdots \\ K_{n+L-1}\left(z,\xi_L^*\right) & K_{n+L-1}\left(\xi_1,\xi_L^*\right) & \cdots & K_{n+L-1}\left(\xi_L,\xi_L^*\right) \end{vmatrix}\\
&\times\ \prod_{i=1}^{L}\left(z-\xi_i\right)^{-1}\ \cdot\ \left[\det\ K_{n+L-1}\left(\xi_i,\xi_j^*\right)\right]^{-1} \quad (16)
\end{aligned}
$$

This result has been obtained by Akemann and Vernizzi in Ref. [4] (in their work $K_n\left(z,\xi^*\right)$ is called $K_{n+1}\left(z,\overline{\xi}\right)$).

The norm $k_n\left(\xi_i;\xi_i^*\right)$ in (9) is proved to be

$$k_n\left(\xi_i;\xi_i^*\right)=h_{n+L}\ \frac{\det\ K_{n+L}\left(\xi_i,\xi_j^*\right)}{\det\ K_{n+L-1}\left(\xi_i,\xi_j^*\right)} \quad (17)$$

as conjectured by Akemann and Vernizzi [4] who verified this formula up to $L = 5$. The results of this section are generalized below to biorthogonal polynomials and all proofs in Sect. 4 and 5 are performed in that case.

2. Biorthogonal Polynomials

Given two set of external sources $\left(\eta_1, ..., \eta_{L_1}\right)$ and $\left(\xi_1, ..., \xi_{L_2}\right)$, we generalize the above constructions and define two sets of biorthogonal monic polynomials $q_n\left(z; \xi_i; \eta_i^*\right)$ and $q_n\left(z; \eta_i; \xi_i^*\right)$ satisfying the orthogonality relation

$$
\int \int d^2z \ q_m^*\left(z; \eta_i; \xi_i^*\right) \ q_n\left(z; \xi_i; \eta_i^*\right) \ \prod_{i=1}^{L_1}\left(z^* - \eta_i^*\right) \prod_{i=1}^{L_2}\left(z - \xi_i\right) \ e^{-V(z, z^*)}
$$
$$
= \ k_n\left(\xi_i; \eta_i^*\right) \ \delta_{nm} \tag{18}
$$

For simplicity, we use the same symbol q_n for the polynomials $q_n\left(z; \xi_i; \eta_i^*\right)$ and $q_n\left(z; \eta_i; \xi_i^*\right)$ although they are different functions depending of the number of variables η_i and ξ_i. The pseudonorms $k_n\left(\xi_i; \eta_i^*\right)$ are complex and may vanish on some manifolds of the $\left(\xi_i; \eta_i^*\right)$ space. The uniqueness of the biorthogonal polynomials supposes that we consider generic values of the sources ξ_i and η_i^* where the norms do not vanish.

The polynomials $q_n\left(z; \xi_i; \eta_i^*\right)$ are naturally given by

$$
q_n\left(z; \xi_i; \eta_i^*\right) = \begin{vmatrix} Q_{n+L_2}\left(z; \eta_i^*\right) & Q_{n+L_2}\left(\xi_1; \eta_i^*\right) & ... & Q_{n+L_2}\left(\xi_{L_2}; \eta_i^*\right) \\ ... & ... & ... & ... \\ Q_{n+1}\left(z; \eta_i^*\right) & Q_{n+1}\left(\xi_1; \eta_i^*\right) & ... & Q_{n+1}\left(\xi_{L_2}; \eta_i^*\right) \\ Q_n\left(z; \eta_i^*\right) & Q_n\left(\xi_1; \eta_i^*\right) & ... & Q_n\left(\xi_{L_2}; \eta_i^*\right) \end{vmatrix}
$$
$$
\times \ \prod_{i=1}^{L_2}\left(z - \xi_i\right)^{-1} \ . \ \left[< n, L_2 >\left(\xi_i; \eta_i^*\right)\right]^{-1} \tag{19}
$$

and correspondingly, the polynomials $q_n\left(z; \eta_i; \xi_i^*\right)$ are obtained from the polynomials $q_n\left(z; \xi_i; \eta_i^*\right)$ by exchanging the ξ's and the η's, an operation which implies the exchange of L_2 and L_1. In (19), $< n, L_2 >\left(\xi_i; \eta_i^*\right)$ is defined as

$$
< n, L_2 >\left(\xi_i; \eta_i^*\right) = \begin{vmatrix} Q_{n+L_2-1}\left(\xi_1; \eta_i^*\right) & ... & Q_{n+L_2-1}\left(\xi_{L_2}; \eta_i^*\right) \\ ... & ... & ... \\ Q_n\left(\xi_1; \eta_i^*\right) & ... & Q_n\left(\xi_{L_2}; \eta_i^*\right) \end{vmatrix} \tag{20}
$$

The polynomials $Q_n\left(z; \eta_i^*\right)$ and $Q_n\left(z; \xi_i^*\right)$ are respectively defined by the following properties: for $m < n$

$$
\int \int d^2z \ p_m^*\left(z\right) \ Q_n\left(z; \xi_i^*\right) \ \prod_{i=1}^{L_2}\left(z^* - \xi_i^*\right) \ e^{-V(z, z^*)} \ = \ 0 \tag{21}
$$

$$
\int \int d^2z \ p_m^*\left(z\right) \ Q_n\left(z; \eta_i^*\right) \ \prod_{i=1}^{L_1}\left(z^* - \eta_i^*\right) \ e^{-V(z, z^*)} \ = \ 0 \tag{22}
$$

5

and the corresponding complex conjugate properties. Again, by simplicity, we use the same notation Q_n in (21) and (22) although they are different functions depending of the number of variables η_i and ξ_i.

The proof of the orthogonality of the polynomials q follows the corresponding proof for the potentials of one variable; the existence, the construction and the properties of the polynomials Q are given in Sect. 4. The polynomials $q_n\left(z;\xi_i;\eta_i^*\right)$ take two different forms depending of the relative values of L_1 and L_2; if $L_1 \leq L_2$

$$
q_n\left(z;\xi_i;\eta_i^*\right)
$$

$$
= \begin{vmatrix}
p_{n+L_2}\left(z\right) & p_{n+L_2}\left(\xi_1\right) & \cdots & p_{n+L_2}\left(\xi_{L_2}\right) \\
\cdots & \cdots & \cdots & \cdots \\
p_{n+L_1}\left(z\right) & p_{n+L_1}\left(\xi_1\right) & \cdots & p_{n+L_1}\left(\xi_{L_2}\right) \\
K_{n+L_1-1}\left(z,\eta_1^*\right) & K_{n+L_1-1}\left(\xi_1,\eta_1^*\right) & \cdots & K_{n+L_1-1}\left(\xi_{L_2},\eta_1^*\right) \\
\cdots & \cdots & \cdots & \cdots \\
K_{n+L_1-1}\left(z,\eta_{L_1}^*\right) & K_{n+L_1-1}\left(\xi_1,\eta_{L_1}^*\right) & \cdots & K_{n+L_1-1}\left(\xi_{L_2},\eta_{L_1}^*\right)
\end{vmatrix}
$$

$$
\times \prod_{i=1}^{L_2}\left(z-\xi_i\right)^{-1} \quad . \quad \left[D_{n+L_1-1}\left(\xi_i;\eta_i^*\right)\right]^{-1} \tag{23}
$$

and if $L_1 \geq L_2 + 1$

$$
q_n\left(z;\xi_i;\eta_i^*\right) = \frac{h_{n+L_2}}{\displaystyle\prod_{i=1}^{L_2}\left(z-\xi_i\right)}
\begin{vmatrix}
p_{n+L_1-1}^*\left(\eta_1\right) & \cdots & p_{n+L_1-1}^*\left(\eta_{L_1}\right) \\
\cdots & \cdots & \cdots \\
p_{n+L_2+1}^*\left(\eta_1\right) & \cdots & p_{n+L_2+1}^*\left(\eta_{L_1}\right) \\
K_{n+L_2}\left(z,\eta_1^*\right) & \cdots & K_{n+L_2}\left(z,\eta_{L_1}^*\right) \\
K_{n+L_2}\left(\xi_1,\eta_1^*\right) & \cdots & K_{n+L_2}\left(\xi_1,\eta_{L_1}^*\right) \\
\cdots & \cdots & \cdots \\
K_{n+L_2}\left(\xi_{L_2},\eta_1^*\right) & \cdots & K_{n+L_2}\left(\xi_{L_2},\eta_{L_1}^*\right)
\end{vmatrix}
$$

$$
\times \frac{1}{D_{n+L_2-1}^*\left(\eta_i;\xi_i^*\right)} \tag{24}
$$

where the determinant $D_n\left(\xi_i;\eta_i^*\right)$ is defined for $L_1 \leq L_2$ and becomes $D_n^*\left(\eta_i;\xi_i^*\right)$ for $L_1 \geq L_2$. In (23),

$$
D_n\left(\xi_i;\eta_i^*\right) = \begin{vmatrix}
p_{n+L_2-L_1}\left(\xi_1\right) & \cdots & p_{n+L_2-L_1}\left(\xi_{L_2}\right) \\
\cdots & \cdots & \cdots \\
p_{n+1}\left(\xi_1\right) & \cdots & p_{n+1}\left(\xi_{L_2}\right) \\
K_n\left(\xi_1,\eta_1^*\right) & \cdots & K_n\left(\xi_{L_2},\eta_1^*\right) \\
\cdots & \cdots & \cdots \\
K_n\left(\xi_1,\eta_{L_1}^*\right) & \cdots & K_n\left(\xi_{L_2},\eta_{L_1}^*\right)
\end{vmatrix} \quad \text{for} \quad L_1 \leq L_2 \tag{25}
$$

Clearly, any line of K_n can be replaced by the corresponding line of K_{n+q} with $0 \leq q \leq L_2 - L_1$. We show in part 2 of the appendix that $D_n\left(\xi_i;\eta_i^*\right) = 0$ for $n < L_1 - 1$ and that $D_{L_1-1}\left(\xi_i;\eta_i^*\right) = \prod_{k=0}^{L_1-1} \frac{1}{h_k} \; \Delta^*\left(\eta_i\right) \; \Delta\left(\xi_i\right)$ where Δ is the

Vandermonde determinant defined in (34). Also in (24),

$$
D_n^* (\eta_i; \xi_i^*) = \begin{vmatrix} p_{n+L_1-L_2}^* (\eta_1) & ... & p_{n+L_1-L_2}^* (\eta_{L_1}) \\ ... & ... & ... \\ p_{n+1}^* (\eta_1) & ... & p_{n+1}^* (\eta_{L_1}) \\ K_n (\xi_1, \eta_1^*) & ... & K_n (\xi_1, \eta_{L_1}^*) \\ ... & ... & ... \\ K_n (\xi_{L_2}, \eta_1^*) & ... & K_n (\xi_{L_2}, \eta_{L_1}^*) \end{vmatrix} \quad \text{for} \quad L_1 \geq L_2 \quad (26)
$$

Again, any line of K_n can be replaced by the corresponding line of K_{n+q} with $0 \leq q \leq L_1 - L_2$. Similarly, we have $D_n^* (\eta_i; \xi_i^*) = 0$ for $n < L_2 - 1$ and

$$
D_{L_2-1}^* (\eta_i; \xi_i^*) = \prod_{k=0}^{L_2-1} \frac{1}{h_k} \ \Delta^* (\eta_i) \ \Delta (\xi_i) .
$$

The pseudonorm $k_n (\xi_i; \eta_i^*)$ is found to be

$$
k_n (\xi_i; \eta_i^*) = (-)^{L_1+L_2} \ h_{n+L_1} \ \frac{D_{n+L_1} (\xi_i; \eta_i^*)}{D_{n+L_1-1} (\xi_i; \eta_i^*)} \quad \text{for } L_1 \leq L_2 \quad (27)
$$

$$
k_n (\xi_i; \eta_i^*) = (-)^{L_1+L_2} \ h_{n+L_2} \ \frac{D_{n+L_2}^* (\eta_i; \xi_i^*)}{D_{n+L_2-1}^* (\eta_i; \xi_i^*)} \quad \text{for } L_1 \geq L_2 \quad (28)
$$

If $L_1 = L_2$ we recover the result (17) with the ξ_j^* replaced by the η_j^*.

In Sect. 6 we apply the above formalism to the calculation of the integrals of the form

$$
\begin{aligned}
& I_{N,L_2,L_1} (\xi_b; \eta_a^*) \\
& = \int \prod_{i=1}^{N} d^2 z_i \prod_{i<j} |z_i - z_j|^2 \prod_{i,a,b} (z_i^* - \eta_a^*) (z_i - \xi_b) \ e^{-\sum_{i=1}^{N} V(z_i, z_i^*)} \quad (29)
\end{aligned}
$$

where i and j run from 1 to N, a from 1 to L_1 and b from 1 to L_2. These integrals are the generalization to potentials of two variables of similar integrals with potentials of one variable which have been expressed by Brezin and Hikami [9] as determinants of the corresponding orthogonal polynomials. The expressions (31-32) have been obtained before by Akemann and Vernizzi [4].

It is easily proved, from the property of the Vandermonde determinant (34) and from the biorthogonality of the polynomials $q_n (z; \xi_i; \eta_i^*)$, that

$$
I_{N,L_2,L_1} (\xi_b; \eta_a^*) = N! \prod_{n=0}^{N-1} k_n (\xi_b; \eta_a^*) \quad (30)
$$

7

This result can be transformed in terms of the determinants $D_n\left(\xi_b; \eta_a^*\right)$ for $L_1 \leq L_2$

$$I_{N,L_2,L_1}\left(\xi_b; \eta_a^*\right) = \frac{(-)^{N(L_1+L_2)} \; N!}{\Delta^*\left(\eta_a\right) \; \Delta\left(\xi_b\right)} \; \prod_{i=0}^{N+L_1-1} h_i \quad D_{N+L_1-1}\left(\xi_b; \eta_a^*\right) \qquad (31)$$

and $D_n^*\left(\eta_a; \xi_b^*\right)$ for $L_1 \geq L_2$

$$I_{N,L_2,L_1}\left(\xi_b; \eta_a^*\right) = \frac{(-)^{N(L_1+L_2)} \; N!}{\Delta^*\left(\eta_a\right) \; \Delta\left(\xi_b\right)} \; \prod_{i=0}^{N+L_2-1} h_i \quad D_{N+L_2-1}^*\left(\eta_a; \xi_b^*\right) \qquad (32)$$

3. A Generalization of the Schur Polynomials

We generalize the definition of the Schur polynomials of several variables $(x_1, ..., x_n)$ to a set of orthogonal, symmetric and non homogeneous polynomials of several variables attached to Young tableaux. We define

$$P_\lambda\left(x_i\right) = \frac{\det \; p_{\lambda_i+n-i}\left(x_j\right)}{\Delta\left(x_i\right)} \qquad (33)$$

where λ is a Young tableau described by the length of its rows $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\Delta\left(x_i\right)$ is the Vandermonde determinant corresponding to the empty Young tableau

$$\Delta\left(x_i\right) = \prod_{i<j}\left(x_i - x_j\right) = \begin{vmatrix} \pi_{n-1}\left(x_1\right) & ... & \pi_{n-1}\left(x_n\right) \\ ... & ... & ... \\ \pi_0\left(x_1\right) & ... & \pi_0\left(x_n\right) \end{vmatrix} \qquad (34)$$

for any set of monic polynomials $\{\pi_n\left(x\right)\}$. In the case where the polynomials $p_n\left(z\right)$ are Ginibre's polynomials, the polynomials $P_\lambda\left(z_i\right)$ are Schur polynomials.

The orthogonality of these polynomials is expressed as

$$\int ... \int \prod_{i=1}^{N} d^2 z_i \; \prod_{i<j}\left|z_i - z_j\right|^2 \; P_\mu^*\left(z_i\right) \; P_\nu\left(z_i\right) \; e^{-\sum_{i=1}^{N} V\left(z_i, z_i^*\right)}$$

$$= N! \; \prod_{i=1}^{N} h_{\lambda_i+N-i} \; \delta_{\mu\nu} \qquad (35)$$

An important relation proved in Sect. 6 is

$$\prod_{i=1}^{N}\prod_{a=1}^{L}\left(z_i - \xi_a\right) = \sum_{\lambda \in [N \times L]} (-)^{NL-|\lambda|} \; P_\lambda\left(z_i\right) \; P_{\widetilde{\lambda'}}\left(\xi_a\right) \qquad (36)$$

where the Young tableau λ' is obtained from the Young tableau λ by exchanging the rows and the columns, and $\widetilde{\lambda'}$ is the complementary Youg tableau of λ' in the rectangle $[L \times N]$; the surface of the Young tableau λ is denoted by $|\lambda|$.

Most of the results of the preceding sections can be expressed in terms of the generalized Schur polynomials. For instance

$$[n, L](\xi_i) = \Delta(\xi_i) \; P_{[L \times n]}(\xi_i) \tag{37}$$

where $[L \times n]$ is the Young tableau corresponding to the rectangle with L rows and n columns. Also the polynomials $Q_n(z; \eta_i^*)$ are found to be

$$Q_n(z; \eta_i^*) = \frac{h_n}{P_{[L \times n]}^*(\eta_i)} \; \sum_{j=0}^{n} \frac{1}{h_j} \; P_j(z) \; P_{[L \times n]_j}^*(\eta_i) \tag{38}$$

where the Young tableau $[L \times n]_j$ is the union of the rectangle $[L - 1 \times n]$ and of the smallest row $\lambda_L = j$. Now, from (62-63) and part 2 of the appendix, we get for $L_1 \le L_2$

$$< \; n, L_2 > (\xi_i; \eta_i^*) = \frac{\Delta(\xi_i)}{P_{[L_1 \times n]}^*(\eta_i)} \prod_{i=0}^{L_1-1} h_{n+i}$$

$$\times \sum_{\lambda \in [L_1 \times n]} \prod_{i=1}^{L_1} \frac{1}{h_{\lambda_i + L_1 - i}} \; P_\lambda^*(\eta_i) \; P_{[\{(L_2 - L_1) \times n\} \cup \lambda]}(\xi_i) \tag{39}$$

and for $L_1 \ge L_2$

$$< \; n, L_2 > (\xi_i; \eta_i^*) = \frac{\Delta(\xi_i)}{P_{[L_1 \times n]}^*(\eta_i)} \prod_{i=0}^{L_2-1} h_{n+i}$$

$$\times \sum_{\lambda \in [L_2 \times n]} \prod_{i=1}^{L_2} \frac{1}{h_{\lambda_i + L_2 - i}} \; P_{[\{(L_1 - L_2) \times n\} \cup \lambda]}^*(\eta_i) \; P_\lambda(\xi_i) \tag{40}$$

The polynomials $q_n(z; \xi_i; \eta_i^*)$ can also be expressed as the ratio of two such expressions; for $L_1 \le L_2$

$$q_n(z; \xi_i; \eta_i^*) = \frac{\sum_{\lambda \in [L_1 \times n]} \prod_{i=1}^{L_1} \frac{1}{h_{\lambda_i + L_1 - i}} \; P_\lambda^*(\eta_i) \; P_{[\{(L_2 - L_1 + 1) \times n\} \cup \lambda]}(z, \xi_i)}{\sum_{\lambda \in [L_1 \times n]} \prod_{i=1}^{L_1} \frac{1}{h_{\lambda_i + L_1 - i}} \; P_\lambda^*(\eta_i) \; P_{[\{(L_2 - L_1) \times n\} \cup \lambda]}(\xi_i)} \tag{41}$$

and for $L_1 \ge L_2 + 1$

$$q_n(z; \xi_i; \eta_i^*) = h_{n+L_2} \frac{A}{B}$$

$$A = \sum_{\lambda \in [(L_2+1) \times n]} \prod_{i=1}^{L_2+1} \frac{1}{h_{\lambda_i + L_2 + 1 - i}} P_{[\{(L_1 - L_2 - 1) \times n\} \cup \lambda]}^*(\eta_i) \; P_\lambda(z, \xi_i)$$

9

$$B = \sum_{\lambda \in [L_2 \times n]} \prod_{i=1}^{L_2} \frac{1}{h_{\lambda_i + L_2 - i}} \; P^*_{[\{(L_1 - L_2) \times n\} \cup \lambda]}(\eta_i) \; P_\lambda(\xi_i)$$

$$(42)$$

In (41) and (42) the denominators make the polynomials $q_n(z; \xi_i; \eta_i^*)$ monic. Similarly, the pseudonorm $k_n(\xi_i; \eta_i^*)$ becomes for $L_1 \leq L_2$

$$k_n(\xi_i; \eta_i^*) = (-)^{L_1 + L_2} \; h_{n + L_1} \; \frac{C}{D}$$

$$C = \sum_{\lambda \in [L_1 \times (n+1)]} \prod_{i=1}^{L_1} \frac{1}{h_{\lambda_i + L_1 - i}} \; P^*_\lambda(\eta_i) \; P_{[\{(L_2 - L_1) \times (n+1)\} \cup \lambda]}(\xi_i)$$

$$D = \sum_{\lambda \in [L_1 \times n]} \prod_{i=1}^{L_1} \frac{1}{h_{\lambda_i + L_1 - i}} \; P^*_\lambda(\eta_i) \; P_{[\{(L_2 - L_1) \times n\} \cup \lambda]}(\xi_i) \qquad (43)$$

and for $L_1 \geq L_2$

$$k_n(\xi_i; \eta_i^*) = (-)^{L_1 + L_2} \; h_{n + L_2} \; \frac{E}{F}$$

$$E = \sum_{\lambda \in [L_2 \times (n+1)]} \prod_{i=1}^{L_2} \frac{1}{h_{\lambda_i + L_2 - i}} \; P^*_{[\{(L_1 - L_2) \times (n+1)\} \cup \lambda]}(\eta_i) \; P_\lambda(\xi_i)$$

$$F = \sum_{\lambda \in [L_2 \times n]} \prod_{i=1}^{L_2} \frac{1}{h_{\lambda_i + L_2 - i}} \; P^*_{[\{(L_1 - L_2) \times n\} \cup \lambda]}(\eta_i) \; P_\lambda(\xi_i) \qquad (44)$$

Finally, we can also express the integrals (29) in terms of the generalized Schur polynomials; from (31-32) and part 2 of the appendix we obtain for $L_1 \geq L_2$

$$I_{N, L_2, L_1}(\xi_b; \eta_a^*) = (-)^{N(L_1 + L_2)} \; N! \; \prod_{i=0}^{N + L_2 - 1} h_i$$

$$\times \sum_{\lambda \in [L_2 \times N]} \prod_{i=1}^{L_2} \frac{1}{h_{\lambda_i + L_2 - i}} \; P^*_{[\{(L_1 - L_2) \times N\} \cup \lambda]}(\eta_a) \; P_\lambda(\xi_b)$$

$$(45)$$

and for $L_1 \leq L_2$

$$I_{N, L_2, L_1}(\xi_b; \eta_a^*) = (-)^{N(L_1 + L_2)} \; N! \; \prod_{i=0}^{N + L_1 - 1} h_i$$

$$\times \sum_{\lambda \in [L_1 \times N]} \prod_{i=1}^{L_1} \frac{1}{h_{\lambda_i + L_1 - i}} \; P^*_\lambda(\eta_a) \; P_{[\{(L_2 - L_1) \times N\} \cup \lambda]}(\xi_b)$$

10

4. The Polynomials $Q_n\left(z; \xi_i^*\right)$

In this section, we look for monic polynomials $Q_n\left(z; \xi_i^*\right)$ satisfying the condition (12). We first show that the polynomials defined in (15) satisfy the condition (12); then, we prove that this solution is unique. First, we define the matrix elements

$$A_{n,m}\left(\xi_i^*\right) = \int\int d^2z \; p_m^*\left(z\right) \; Q_n\left(z; \xi_i^*\right) \; e^{-V\left(z, z^*\right)} \tag{47}$$

which are zero for $m > n$. From (15) we obtain

$$A_{n,m}\left(\xi_i^*\right) = \frac{h_n}{[n,L]^*\left(\xi_i\right)} \begin{vmatrix} p_{n+L-1}^*\left(\xi_1\right) & \cdots & p_{n+L-1}^*\left(\xi_L\right) \\ \cdots & \cdots & \cdots \\ p_{n+1}^*\left(\xi_1\right) & \cdots & p_{n+1}^*\left(\xi_L\right) \\ p_m^*\left(\xi_1\right) & \cdots & p_m^*\left(\xi_L\right) \end{vmatrix} \quad \text{for } m < n + L \tag{48}$$

Clearly, this determinant vanishes for $n < m < n+L$. However, it shows that for $m < n + L$ the elements $A_{n,m}\left(\xi_i^*\right)$ are linear combinations of the polynomials $p_m^*\left(\xi_i\right)$, $i = 1, ..., L$ with L coefficients independant of m.

From the property (34) of the Vandermonde determinant we may write

$$p_m\left(z\right)\prod_{i=1}^{L}\left(z - \xi_i\right) = \frac{1}{\Delta\left(r_i, \xi_j\right)}$$

$$\times \begin{vmatrix} p_{m+L}\left(z\right) & p_{m+L}\left(r_1\right) & \cdots & p_{m+L}\left(r_m\right) & p_{m+L}\left(\xi_1\right) & \cdots & p_{m+L}\left(\xi_L\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_0\left(z\right) & p_0\left(r_1\right) & \cdots & p_0\left(r_m\right) & p_0\left(\xi_1\right) & \cdots & p_0\left(\xi_L\right) \end{vmatrix} \tag{49}$$

where $r_1, ..., r_m$ are the roots of the polynomial $p_m\left(z\right)$ and $\Delta\left(r_i, \xi_j\right)$ is the corresponding Vandermonde determinant. We now calculate the integral (12) for $m < n$; we find $\left[\Delta^*\left(r_i, \xi_j\right)\right]^{-1}$ times the determinant

$$\begin{vmatrix} A_{n,m+L}\left(\xi_i^*\right) & p_{m+L}^*\left(r_1\right) & \cdots & p_{m+L}^*\left(r_m\right) & p_{m+L}^*\left(\xi_1\right) & \cdots & p_{m+L}^*\left(\xi_L\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n,0}\left(\xi_i^*\right) & p_0^*\left(r_1\right) & \cdots & p_0^*\left(r_m\right) & p_0^*\left(\xi_1\right) & \cdots & p_0^*\left(\xi_L\right) \end{vmatrix} \tag{50}$$

This determinant is zero for $m < n$ since, for $q < n+L$, the elements $A_{n,q}\left(\xi_i^*\right)$ are linear combinations of the polynomials $p_q^*\left(\xi_j\right)$, $j = 1, ..., L$ with the same q independant coefficients. This achieve the proof of (12) for the polynomials (15).

Before proving the unicity of the polynomials $Q_n\left(z;\xi_i^*\right)$, we prove that the integral

$$\int\int d^2z \ p_n^*\left(z\right) Q_n\left(z;\xi_i^*\right) \prod_{i=1}^{L} \left(z^*-\xi_i^*\right) \ e^{-V(z,z^*)} = (-)^L \ h_n\frac{[n+1,L]^*\left(\xi_i\right)}{[n,L]^*\left(\xi_i\right)}$$

$$(51)$$

This integral is calculated exactly as above with m replaced by n; since $L>0$ we obtain $\left[\Delta^*\left(r_i,\xi_j\right)\right]^{-1}$ times the determinant

$$\begin{vmatrix} 0 & p_{n+L}^*\left(r_1\right) & ... & p_{n+L}^*\left(r_n\right) & p_{n+L}^*\left(\xi_1\right) & ... & p_{n+L}^*\left(\xi_L\right) \\ A_{n,n+L-1}\left(\xi_i^*\right) & p_{n+L-1}^*\left(r_1\right) & ... & p_{n+L-1}^*\left(r_n\right) & p_{n+L-1}^*\left(\xi_1\right) & ... & p_{n+L-1}^*\left(\xi_L\right) \\ ... & ... & ... & ... & ... & ... & ... \\ A_{n,0}\left(\xi_i^*\right) & p_0^*\left(r_1\right) & ... & p_0^*\left(r_n\right) & p_0^*\left(\xi_1\right) & ... & p_0^*\left(\xi_L\right) \end{vmatrix}$$

$$(52)$$

The same combination of the $p_q^*\left(\xi_i\right)$ which makes the $A_{n,q}\left(\xi_i^*\right)$ for $q = 0,...,n+L-1,$ transforms (52) into

$$\begin{vmatrix} B_{n,n+L}\left(\xi_i^*\right) & p_{n+L}^*\left(r_1\right) & ... & p_{n+L}^*\left(r_n\right) & p_{n+L}^*\left(\xi_1\right) & ... & p_{n+L}^*\left(\xi_L\right) \\ 0 & p_{n+L-1}^*\left(r_1\right) & ... & p_{n+L-1}^*\left(r_n\right) & p_{n+L-1}^*\left(\xi_1\right) & ... & p_{n+L-1}^*\left(\xi_L\right) \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & p_0^*\left(r_1\right) & ... & p_0^*\left(r_n\right) & p_0^*\left(\xi_1\right) & ... & p_0^*\left(\xi_L\right) \end{vmatrix}$$

$$(53)$$

which is simply equal to $\Delta^*\left(r_i,\xi_j\right) \ B_{n,n+L}\left(\xi_i^*\right)$ with

$$B_{n,n+L}\left(\xi_i^*\right) = (-)^L \ h_n\frac{[n+1,L]^*\left(\xi_i\right)}{[n,L]^*\left(\xi_i\right)}$$

$$(54)$$

This ends the proof of (51).

We now prove the uniqueness of the polynomials $Q_n\left(z;\xi_i^*\right)$. The proof is a recurrence: $Q_0\left(z;\xi_i^*\right) = 1$; let us suppose that there exists two polynomials $Q_1\left(z;\xi_i^*\right)$ and $Q_1'\left(z;\xi_i^*\right)$, by difference we obtain

$$\int\int d^2z \ \prod_{i=1}^{L} \left(z^*-\xi_i^*\right) \ e^{-V(z,z^*)} = 0$$

$$(55)$$

which, from (51), is equivalent to $[1,L]^*\left(\xi_i\right) = 0$. Of course, this is wrong for generic values of the variables ξ_i.

We suppose now that we proved that the polynomials $Q_k\left(z;\xi_i^*\right)$ are unique for $k = 0,...,n-1$ and consequently given by (15). If there exists two polynomials $Q_n\left(z;\xi_i^*\right)$ and $Q_n'\left(z;\xi_i^*\right)$, then by difference, we have

$$\int\int d^2z \ p_k^*\left(z\right) Q_{n-1}\left(z;\xi_i^*\right) \prod_{i=1}^{L} \left(z^*-\xi_i^*\right) \ e^{-V(z,z^*)} = 0 \quad \text{for} \ \ k = 0,...,n-1$$

$$(56)$$

12

Because of the recurrence, the unique polynomial $Q_{n-1}(z;\xi_i^*)$ which satisfy (56) for $k = 0, ..., n-2$ is given by (15); Consequently, for $k = n-1$ equation (56) is equivalent to $[n, L]^*(\xi_i) = 0$ which is wrong for generic values of the variables ξ_i.

This ends the proof of the uniqueness of the polynomials $Q_n(z;\xi_i^*)$.

We now establish a first relation for the pseudonorm $k_n(\xi_i;\eta_i^*)$. From the definition (18) we easily obtain two relations

$$k_n(\xi_i;\eta_i^*) = (-)^{L_2} \frac{<n+1, L_2 > (\xi_i;\eta_i^*)}{<n, L_2 > (\xi_i;\eta_i^*)}$$

$$\times \int\int d^2z \; p_n^*(z) \; Q_n(z;\eta_i^*) \; \prod_{i=1}^{L_1} (z^* - \eta_i^*) \quad e^{-V(z,z^*)} \quad (57)$$

$$k_n(\xi_i;\eta_i^*) = (-)^{L_1} \frac{<n+1, L_1 >^* (\eta_i;\xi_i^*)}{<n, L_1 >^* (\eta_i;\xi_i^*)}$$

$$\times \int\int d^2z \; Q_n^*(z;\xi_i^*) \; p_n(z) \; \prod_{i=1}^{L_2} (z - \xi_i) \quad e^{-V(z,z^*)} \quad (58)$$

From (51) we may write

$$k_n(\xi_i, \eta_i^*) = (-)^{L_1+L_2} \; h_n \; \frac{[n+1, L_1]^*(\eta_i)}{[n, L_1]^*(\eta_i)}$$

$$\times \frac{<n+1, L_2 > (\xi_i;\eta_i^*)}{<n, L_2 > (\xi_i;\eta_i^*)} \quad (59)$$

$$k_n(\xi_i, \eta_i^*) = (-)^{L_1+L_2} \; h_n \; \frac{[n+1, L_2](\xi_i)}{[n, L_2](\xi_i)}$$

$$\times \frac{<n+1, L_1 >^* (\eta_i;\xi_i^*)}{<n, L_1 >^* (\eta_i;\xi_i^*)} \quad (60)$$

These relations will be transformed further later on.

5. The Polynomials $q_n(z;\xi_i;\eta_i^*)$

In this section, we calculate the determinants $<n, L_2 > (\xi_i;\eta_i^*)$ defined in (20) for the pair of variables $(\eta_1, ..., \eta_{L_1})$ and $(\xi_1, ..., \xi_{L_2})$. From (15), we see that for $L_1 \geq 1$, the polynomials $Q_n(\xi;\eta_i^*)$ can be developed as

$$Q_n(\xi;\eta_i^*) = \frac{h_n}{[n, L_1]^*(\eta_i)} \sum_{j=1}^{L_1} (-)^{L_1-j} \; [n+1, L_1-1]_{\hat{j}}^*(\eta_i) \quad K_q(\xi, \eta_{\hat{j}}^*) \quad (61)$$

for any q such that $n \leq q < n+L_1$. The symbol $[n+1, L_1-1]_{\hat{j}}^*(\eta_i)$ means that the column corresponding to η_j is ommited in the determinant; by convention $[n+1, 0]_{\hat{j}}^*(\eta_i) = 1$.

13

The rather lengthy calculation of $< n, L_2 > (\xi_i; \eta_i^*)$ is given in part 1 of the appendix; the expression of the result depends whether L_1 is larger or not to L_2. We find

$$< \ n, L_2 > (\xi_i; \eta_i^*) = \frac{\prod\limits_{i=0}^{L_1-1} h_{n+i}}{[n, L_1]^* (\eta_i)} \ D_{n+L_1-1} (\xi_i; \eta_i^*) \ \text{for} \ L_2 \geq L_1 \quad (62)$$

$$< \ n, L_2 > (\xi_i; \eta_i^*) = \frac{\prod\limits_{i=0}^{L_2-1} h_{n+i}}{[n, L_1]^* (\eta_i)} \ D_{n+L_2-1}^* (\eta_i; \xi_i^*) \ \text{for} \ L_2 \leq L_1 \quad (63)$$

where the determinants $D_n (\xi_i; \eta_i^*)$ and $D_n^* (\eta_i; \xi_i^*)$ are defined in (25-26).

From (19) we may write the polynomials $q_n (z; \xi_i; \eta_i^*)$ as

$$q_n (z; \xi_i; \eta_i^*) = \frac{1}{\prod\limits_{i=1}^{L_2} (z - \xi_i)} \ \frac{< n, L_2 + 1 > (z, \xi_i; \eta_i^*)}{< n, L_2 > (\xi_i; \eta_i^*)} \quad (64)$$

and from (62-63) we obtain the expressions (23-24). We also get from (59-60) the pseudonorm $k_n (\xi_i, \eta_i^*)$ as in (27-28).

6. Applications

We exploit the existence of the biorthogonal polynomials to calculate the integrals (29). We write

$$\prod_{i<j} |z_i - z_j|^2 = \begin{vmatrix} q_{N-1}^* (z_1; \eta_i; \xi_i^*) & \cdots & q_{N-1}^* (z_N; \eta_i; \xi_i^*) \\ \cdots & \cdots & \cdots \\ q_0^* (z_1; \eta_i; \xi_i^*) & \cdots & q_0^* (z_N; \eta_i; \xi_i^*) \end{vmatrix}$$
$$\times \begin{vmatrix} q_{N-1} (z_1; \xi_i; \eta_i^*) & \cdots & q_{N-1} (z_N; \xi_i; \eta_i^*) \\ \cdots & \cdots & \cdots \\ q_0 (z_1; \xi_i; \eta_i^*) & \cdots & q_0 (z_N; \xi_i; \eta_i^*) \end{vmatrix} \quad (65)$$

Then, we obtain (30) by developing the determinants and by integrating over each variables z_i using the biorthogonality of the polynomials q_n.

Now, we establish the relation (36) between $\prod\limits_{i=1}^{N} \prod\limits_{a=1}^{L} (z_i - \xi_a)$ and the generalized Schur polynomials (33). First, we write

$$\Delta (z_i) \ \Delta (\xi_a) \prod_{i=1}^{N} \prod_{a=1}^{L} (z_i - \xi_a) =$$

14

$$\left| \begin{array}{cccccc} p_{N+L-1}\left(z_1\right) & ... & p_{N+L-1}\left(z_N\right) & p_{N+L-1}\left(\xi_1\right) & ... & p_{N+L-1}\left(\xi_L\right) \\ ... & ... & ... & ... & ... & ... \\ p_0\left(z_1\right) & ... & p_0\left(z_N\right) & p_0\left(\xi_1\right) & ... & p_0\left(\xi_L\right) \end{array} \right| \tag{66}$$

Then, we develop the determinant by separating the z_i part from the ξ_a part. We get

$$\Delta\left(z_i\right)\,\Delta\left(\xi_a\right)\prod_{i=1}^{N}\prod_{a=1}^{L}\left(z_i-\xi_a\right)=$$

$$\sum_{I_N}\left(-\right)^{s\left(I_n\right)}\left| \begin{array}{ccc} p_{i_N}\left(z_1\right) & ... & p_{i_N}\left(z_N\right) \\ ... & ... & ... \\ p_{i_1}\left(z_1\right) & ... & p_{i_1}\left(z_N\right) \end{array} \right|\left| \begin{array}{ccc} p_{j_L}\left(\xi_1\right) & ... & p_{j_L}\left(\xi_L\right) \\ ... & ... & ... \\ p_{j_1}\left(\xi_1\right) & ... & p_{j_1}\left(\xi_L\right) \end{array} \right| \tag{67}$$

where we sum over all subsets of N indices $I_N = \{i_1 < ... < i_N\}$
$\subset \{0, ..., N+L-1\}$ and where $\{j_1 < ... < j_L\}$ is the complementary subset; the signature for the sign is

$$s\left(I_n\right) = \frac{N\left(2L+N-1\right)}{2}-\sum_{a=1}^{N}i_a \tag{68}$$

Now, we define two Young tableaux λ and μ from the length of their rows

$$\begin{aligned} \lambda_{N-a+1} &= i_a - a + 1 & a = 1,...,N \tag{69} \\ \mu_{L-b+1} &= j_b - b + 1 & b = 1,...,L \tag{70} \end{aligned}$$

We observe the relations

$$\sum_{a=1}^{N}i_a + \sum_{b=1}^{L}j_b = \frac{\left(N+L\right)\left(N+L-1\right)}{2} \tag{71}$$

$$|\lambda| = \sum_{a=1}^{N}i_a - \frac{N\left(N-1\right)}{2} \tag{72}$$

$$|\mu| = \sum_{b=1}^{L}j_b - \frac{L\left(L-1\right)}{2} \tag{73}$$

so that $|\lambda| + |\mu| = NL$.

Clearly, the Young tableau λ belongs to the rectangle $[N \times L]$, and the Young tableau μ belongs to the rectangle $[L \times N]$. Now, we show in part 3 of the appendix that the complementarity of the indices $\{i_a\}$ and $\{j_b\}$ makes $\mu = \widetilde{\lambda}'$ as defined in Sect. 3 (36). Consequently, (67) is nothing but (36).

We may now calculate $I_{N,L_2,L_1}\left(\xi_b; \eta_a^*\right)$ from the orthogonality property (35) of the polynomials $P_\lambda\left(z_i\right)$; we obtain

$$I_{N,L_2,L_1}\left(\xi_b; \eta_a^*\right) = \left(-\right)^{N\left(L_1+L_2\right)}\,N!\sum_{\lambda\in[N\times L]}\prod_{i=1}^{N}h_{\lambda_i+N-i}\;P_{\widetilde{\lambda_2'}}^*\left(\xi_b\right)\;P_{\widetilde{\lambda_1'}}\left(\eta_a\right) \tag{74}$$

15

where $\widetilde{\lambda'_1}$ and $\widetilde{\lambda'_2}$ are the complementary Young tableaux of λ' in the rectangles $[L_1 \times N]$ and $[L_2 \times N]$ respectively and where $L = Inf(L_1, L_2)$. We thus obtain (45) or (46) by relabelling $\widetilde{\lambda'_1}$ or $\widetilde{\lambda'_2}$ as λ depending whether $L_1 \geq L_2$ or not.

Both formulas, (45 or 46) and (74), agree because of the following property proved in part 3 of the appendix: given a Young tableau λ in the rectangle $[L \times N]$ and the corresponding Young tableau $\mu = \widetilde{\lambda'}$ in the rectangle $[N \times L]$, then

$$\prod_{i=1}^{L} h_{\lambda_i + L - i} \ \prod_{i=1}^{N} h_{\mu_i + N - i} = \prod_{i=0}^{L+N-1} h_i \tag{75}$$

7. Discussion

Given a positive Borel measure $\mu(x)$ on the real line and its set of orthogonal, monic polynomials $\{p_n(x)\}$ satisfying

$$\int \ p_n(x) \ \ p_m(x) \ \ d\mu(x) = h_n \ \ \delta_{nm} \tag{76}$$

we define two operations:

$$operation \ 1 \quad : \quad d\mu_1(x) = (x - \xi) \ \ d\mu(x) \tag{77}$$

$$operation \ 2 \quad : \quad d\mu_2(x) = \frac{1}{x - y} \ \ d\mu(x) \tag{78}$$

Although $\mu_1(x)$ and $\mu_2(x)$ are not positive Borel measures, operations 1 and 2 transform the set of orthogonal polynomials $p_n(x)$ into new sets of orthogonal monic polynomials defined as

- operation 1:

$$p_n(x) \quad \rightarrow \quad q_n(x) = \frac{1}{x - \xi} \left| \begin{array}{cc} p_{n+1}(x) & p_{n+1}(\xi) \\ p_n(x) & p_n(\xi) \end{array} \right| \ \frac{1}{p_n(\xi)} \tag{79}$$

$$h_n \quad \rightarrow \quad k_n(\xi) = - \ h_n \ \frac{p_{n+1}(\xi)}{p_n(\xi)} \tag{80}$$

- operation 2:

$$p_n(x) \quad \rightarrow \quad q_n(x) = \left| \begin{array}{cc} p_n(x) & h_n(y) \\ p_{n-1}(x) & h_{n-1}(y) \end{array} \right| \ \frac{1}{h_{n-1}(y)} \qquad n > 0 \tag{81}$$

$$h_n \quad \rightarrow \quad k_n(y) = - \ h_{n-1} \ \frac{h_n(y)}{h_{n-1}(y)} \qquad n > 0 \tag{82}$$

where $h_n(y)$ is the Cauchy transform of $p_n(x)$ defined for $y \notin \mathbb{R}$ as

$$h_n(y) = \frac{1}{2i\pi} \int \frac{p_n(x)}{x - y} \ d\mu(x) \tag{83}$$

16

Of course, $q_0(x) = 1$ and $k_0(y) = 2i\pi\, h_0(y)$.

Successive iterations of operation 1 generate Christoffel's results (2-3) and (7). As an application, we obtain Brezin's and Hikami's result [9]

$$\int \ldots \int \prod_{i=1}^{N} d\mu(x_i) \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^{N}\prod_{a=1}^{L} (x_i - \xi_a)$$

$$= N! \prod_{n=0}^{N-1} k_n(\xi_a) = (-)^{LN}\ N! \prod_{n=0}^{N-1} h_n\ \frac{[N,L](\xi_a)}{\Delta(\xi_a)} \tag{84}$$

Successive iterations of operations 1 and 2 generate a set of orthogonal monic polynomials described by Uvarov [10], namely, for a given set of external sources $(\xi_1,...,\xi_L)$ at the numerator and $(y_1,...,y_M)$ at the denominator, we have

$$\int d\mu(x)\ q_n(x)\ q_m(x)\ \frac{\displaystyle\prod_{i=1}^{L}(x-\xi_i)}{\displaystyle\prod_{j=1}^{M}(x-y_j)} = k_n(\xi_i; y_j)\ \delta_{nm} \tag{85}$$

The orthogonal polynomials are, for $n \geq M$

$$q_n(x) = \prod_{i=1}^{L} (x-\xi_i)^{-1}\ \ [[n-M, L+M](\xi_i; y_j)]^{-1}\ \times$$

$$\begin{vmatrix} p_{n+L}(x) & p_{n+L}(\xi_1) & ... & p_{n+L}(\xi_L) & h_{n+L}(y_1) & ... & h_{n+L}(y_M) \\ ... & ... & ... & ... & ... & ... & ... \\ p_{n-M}(x) & p_{n-M}(\xi_1) & ... & p_{n-M}(\xi_L) & h_{n-M}(y_1) & ... & h_{n-M}(y_M) \end{vmatrix} \tag{86}$$

and

$$k_n(\xi_i; y_j) = (-)^{L+M}\ h_{n-M}\ \frac{[n-M+1, L+M](\xi_i; y_j)}{[n-M, L+M](\xi_i; y_j)} \tag{87}$$

where $[n-M, L+M](\xi_i; y_j)$ is the minor of $p_{n+L}(x)$ in the determinant (86). For $n < M$ equations (86-87) are replaced by

$$q_n(x) = \prod_{i=1}^{L} (x-\xi_i)^{-1}\ \ [[n-M, L+M](\xi_i; y_j)]^{-1}\ \times$$

$$\begin{vmatrix} p_{n+L}(x) & p_{n+L}(\xi_1) & ... & p_{n+L}(\xi_L) & h_{n+L}(y_1) & ... & h_{n+L}(y_M) \\ ... & ... & ... & ... & ... & ... & ... \\ p_0(x) & p_0(\xi_1) & ... & p_0(\xi_L) & h_0(y_1) & ... & h_0(y_M) \\ 0 & 0 & ... & 0 & p_{M-n-1}(y_1) & ... & p_{M-n-1}(y_M) \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & p_0(y_1) & ... & p_0(y_M) \end{vmatrix} \tag{88}$$

and

$$k_n\left(\xi_i; y_j\right) = (-)^{L+n} \; 2i\pi \; \frac{[n-M+1, L+M]\left(\xi_i; y_j\right)}{[n-M, L+M]\left(\xi_i; y_j\right)} \tag{89}$$

We find convenient to keep the notation $[n-M, L+M]\left(\xi_i; y_j\right)$ for the minor of $p_{n+L}\left(x\right)$ in the determinant (88) although $(n-M) < 0$; in this notation, $(M-n)$ is the number of lines of zeroes and of $p_q\left(y_a\right)$ and $(L+M)$ is the size of the determinant. As n increases to M the number of lines of zeroes and of $p_q\left(y_a\right)$ decreases and that provides a natural transition from (88) to (86) as n becomes larger than M. As a special case, we have $[-M, L+M]\left(\xi_i; y_j\right) = \Delta\left(\xi_i\right) \; \Delta\left(y_j\right)$.

We note that in Uvarov's expression for (88), $p_q\left(y_a\right)$ for $0 \leq q \leq M-n-1$ is replaced by y_a^q (which is the same situation as in the Vandermonde determinant and leaves the value of the determinant unchanged).

As a consequence of this set of orthogonal polynomials, we can generalize the calculation of the integrals (84) to the integrals

$$I_N = \int \ldots \int \prod_{i=1}^{N} d\mu\left(x_i\right) \prod_{i<j}\left(x_i - x_j\right)^2 \frac{\displaystyle\prod_{i=1}^{N}\prod_{a=1}^{L}\left(x_i - \xi_a\right)}{\displaystyle\prod_{i=1}^{N}\prod_{b=1}^{M}\left(x_i - y_b\right)} = N! \prod_{n=0}^{N-1} k_n\left(\xi_a; y_b\right) \tag{90}$$

For $N > M$, equation (90) is

$$I_N = N! \; (-)^{(L+M)N} \; (-)^{\frac{M(M+1)}{2}} \; (2i\pi)^M \; \frac{\displaystyle\prod_{i=0}^{N-M-1} h_i}{\Delta\left(\xi_a\right) \; \Delta\left(y_b\right)} \; [N-M, L+M]\left(\xi_i; y_j\right) \tag{91}$$

this result has been obtained by Fyodorov and Strahov [11]; for $N \leq M$, we obtain

$$I_N = N! \; (-)^{LN} \; (-)^{\frac{N(N-1)}{2}} \; (2i\pi)^N \; \frac{1}{\Delta\left(\xi_a\right) \; \Delta\left(y_b\right)} \; [N-M, L+M]\left(\xi_i; y_j\right) \tag{92}$$

Of course, when some ξ's and some y's are equal, equations (91-92) provide many relations between determinants. Moreover, if $L = M$, it is shown in the appendix (part 4) that equations (91-92) can be expressed in terms of the determinant of a $[M \times M]$ matrix; for $N \geq M$,

$$\begin{aligned}
I_N &= N! \; (-)^{\frac{M(M-1)}{2}} \left(\prod_{i=0}^{N-1} h_i\right) \frac{\displaystyle\prod_{a=1}^{M}\prod_{b=1}^{M}\left(y_b - \xi_a\right)}{\Delta\left(\xi_a\right) \; \Delta\left(y_b\right)} \\
&\quad \times \det\left[2i\pi \; H_{N-1}\left(\xi_a; y_b\right) + \frac{1}{y_b - \xi_a}\right]
\end{aligned} \tag{93}$$

18

where

$$H_N\left(\xi_a; y_b\right) = \sum_{i=0}^{N} \frac{1}{h_i}\ p_i\left(\xi_a\right)\ h_i\left(y_b\right) \tag{94}$$

Similarly, for $N \le M$ it is shown in part 4 of the appendix that

$$I_N = N!\ \left(-\right)^{\frac{N(N-1)}{2}} \frac{\displaystyle\prod_{i=0}^{N-1} h_i}{\Delta\left(\theta_a\right)\ \Delta\left(y_b\right)} \left| \begin{array}{ccc} U_{N-1}\left(\theta_1, y_1\right) & \dots & U_{N-1}\left(\theta_1, y_M\right) \\ \dots & \dots & \dots \\ U_{N-1}\left(\theta_N, y_1\right) & \dots & U_{N-1}\left(\theta_N, y_M\right) \\ p_{M-N-1}\left(y_1\right) & \dots & p_{M-N-1}\left(y_M\right) \\ \dots & \dots & \dots \\ p_0\left(y_1\right) & \dots & p_0\left(y_M\right) \end{array} \right| \tag{95}$$

where

$$U_{N-1}\left(\theta_a, y_j\right) = \prod_{i=1}^{M}\left(y_j - \xi_i\right)\ \left[2i\pi\ H_{N-1}\left(\theta_a; y_j\right) + \frac{1}{y_j - \theta_a}\right] \tag{96}$$

and where the set of N variables θ_a is any subset of $\left(\xi_1, ..., \xi_M\right)$. If $M = N$ there is no more $p_q\left(y_j\right)$ part in (95) and we recover(93).

Now, if we take in (90) and (93) the derivatives of I_N in regards to all variables ξ_a, then, if we make all variables $\xi_a = y_a$, we prove in part 4 of the appendix that for $N \ge M$

$$\begin{aligned} J_N &= \int \dots \int \prod_{i=1}^{N} d\mu\left(x_i\right) \prod_{i<j}\left(x_i - x_j\right)^2 \prod_{a=1}^{M}\left(\sum_{i=1}^{N} \frac{1}{x_i - \xi_a}\right) \\ &= N!\ \prod_{i=0}^{N-1} h_i\ \ "\det" T_M\left(\xi_a; \xi_b\right) \end{aligned} \tag{97}$$

where $T_M\left(\xi_a; \xi_b\right)$ is a $[M \times M]$ matrix, the elements of which are

$$\begin{aligned} \left[T_M\right]_{aa} &= 2i\pi\ H_{N-1}\left(\xi_a; \xi_a\right) \tag{98} \\ \left[T_M\right]_{a \ne b} &= 2i\pi\ H_{N-1}\left(\xi_a; \xi_b\right) + \frac{1}{\xi_b - \xi_a} \tag{99} \end{aligned}$$

and where "det" is a notation which means that we ignore the double poles at $\xi_a = \xi_b$ as we develop the determinant. Moreover, it is easy to see that the residues of the single poles at $\xi_a = \xi_b$ are zero, so that the integrals (97) are analytic in ξ's if the ξ's $\notin \mathbb{R}$.

The discontinuity of $h_n\left(y\right)$ when $y \in \mathbb{R}$ is given by (83)

$$disc\ h_n\left(y\right) = p_n\left(y\right)\ \mu'\left(y\right) \tag{100}$$

19

where we use the notation $d\mu(y) = \mu'(y) \ dy$ .Consequently, for $N \geq M$ the total discontinuity of J_N , for a set of different real variables ξ's (which is related to the N-point correlation function), is

$$disc_{\xi_1,...,\xi_M} \ J_N = N! \ \left(\prod_{i=0}^{N-1} h_i\right) \ (2i\pi)^M \ \left(\prod_{i=1}^{M} \mu'(\xi_i)\right) \det \ K_{N-1}(\xi_a;\xi_b) \tag{101}$$

as already obtained in Ref. $[12-13]$. For $N < M$ the expression for J_N is more complicate but the total discontinuity in that case is clearly zero.

A similar program should be achieved for potentials of two variables $V(z, z^*)$.

We consider a positive Borel measure $\mu(z, z^*)$ on the complex plane and its set of orthogonal, monic polynomials $\{p_n(z)\}$ satisfying

$$\int \ p_n^*(z) \ p_m(z) \ d\mu(z, z^*) = \ h_n \ \delta_{nm} \tag{102}$$

We define four operations:

$$operation 1 \ : \ d\mu_1(z, z^*) = (z - \xi) \ d\mu(z, z^*) \tag{103}$$
$$operation 2 \ : \ d\mu_2(z, z^*) = (z^* - \eta^*) \ d\mu(z, z^*) \tag{104}$$
$$operation 3 \ : \ d\mu_3(z, z^*) = \frac{1}{(z - y)} \ d\mu(z, z^*) \tag{105}$$
$$operation \ 4 \ : \ d\mu_4(z, z^*) = \frac{1}{(z^* - x^*)} \ d\mu(z, z^*) \tag{106}$$

Operations 1 and 2 transform the set of orthogonal polynomials $p_n(z)$ into sets of biorthogonal monic polynomials defined as
- operation 1:

$$p_n(z) \ \rightarrow \ q_n(z; \xi; \Phi) = \frac{1}{(z - \xi)} \begin{vmatrix} p_{n+1}(z) & p_{n+1}(\xi) \\ p_n(z) & p_n(\xi) \end{vmatrix} \ \frac{1}{p_n(\xi)} \tag{107}$$

$$p_n^*(z) \ \rightarrow \ q_n^*(z; \Phi; \xi^*) = Q_n^*(z; \xi^*) = \frac{h_n}{p_n(\xi)} \ K_n^*(z, \xi^*) \tag{108}$$

$$h_n \ \rightarrow \ k_n(\xi) = -h_n \frac{p_{n+1}(\xi)}{p_n(\xi)} \tag{109}$$

In (107-108), Φ means an empty set of variables, Q_n and K_n are defined in (13-14) and have the property that

$$\int Q_n^*(z; \xi^*) \ p_m(z) \ (z - \xi) \ d\mu(z, z^*) = 0 \quad \text{for} \ \ m < n \tag{110}$$

20

- operation 2:

$$p_n(z) \quad \rightarrow \quad q_n(z; \Phi; \eta^*) = Q_n(z; \eta^*) = \frac{h_n}{p_n^*(\eta)} \, K_n(z, \eta^*) \tag{111}$$

$$p_n^*(z) \quad \rightarrow \quad q_n^*(z; \eta; \Phi) = \frac{1}{(z^* - \eta^*)} \begin{vmatrix} p_{n+1}^*(z) & p_{n+1}^*(\eta) \\ p_n^*(z) & p_n^*(\eta) \end{vmatrix} \frac{1}{p_n^*(\eta)} \tag{112}$$

$$h_n \quad \rightarrow \quad k_n(\eta^*) = -h_n \frac{p_{n+1}^*(\eta)}{p_n^*(\eta)} \tag{113}$$

where, similarly to (110), we have

$$\int p_m^*(z) \quad Q_n(z; \eta^*) \quad (z^* - \eta^*) \quad d\mu(z, z^*) = 0 \quad \text{for} \quad m < n \tag{114}$$

Successive iterations of operations 1 and 2 is the purpose of this publication and generate the biorthogonal polynomials $q_n(z; \xi_i; \eta_i^*)$ and $q_n^*(z; \eta_i; \xi_i^*)$ described in (18-19). Operations 3 and 4 are beyond the scope of this publication and might be described in a near future.

Appendix

Part 1: Given two sets of source variables $(\eta_1, ..., \eta_{L_1})$ and $(\xi_1, ..., \xi_{L_2})$, this part of the appendix is devoted to the calculation of $< n, L_2 > (\xi_i; \eta_i^*)$ defined in (20).

1°) We first suppose that $L_2 \geq L_1$. From (20) and (61) we write

$$\begin{aligned}
< \quad n, L_2 > (\xi_i; \eta_i^*) = & \prod_{i=0}^{L_2-1} \frac{h_{n+i}}{[n+i, L_1]^*(\eta_i)} \\
& \times \sum_{\{j_k\}=1}^{L_1} (-)^{L_1 L_2 - \sum j_k} \prod_{k=1}^{L_2} [n+1+L_2-k, L_1-1]_{\hat{j_k}}^*(\eta_i) \\
& \times \begin{vmatrix}
K_{n+L_2-1}\left(\xi_1, \eta_{j_1}^*\right) & ... & K_{n+L_2-1}\left(\xi_{L_2}, \eta_{j_1}^*\right) \\
... & ... & ... \\
K_{n+L_1}\left(\xi_1, \eta_{j_{L_2-L_1}}^*\right) & ... & K_{n+L_1}\left(\xi_{L_2}, \eta_{j_{L_2-L_1}}^*\right) \\
K_{n+L_1-1}\left(\xi_1, \eta_{j_{L_2-L_1+1}}^*\right) & ... & K_{n+L_1-1}\left(\xi_{L_2}, \eta_{j_{L_2-L_1+1}}^*\right) \\
... & ... & ... \\
K_{n+L_1-1}\left(\xi_1, \eta_{j_{L_2}}^*\right) & ... & K_{n+L_1-1}\left(\xi_{L_2}, \eta_{j_{L_2}}^*\right)
\end{vmatrix} \quad (A1)
\end{aligned}$$

where we sum over all indices $\{j_1, ..., j_{L_2}\}$ running from 1 to L_1. In the L_1 bottom lines of the determinant, the η_i^* variables in the functions K_{n+L_1-1} take all possible values $\eta_1^*, ..., \eta_{L_1}^*$. Consequently, $K_{n+L_1}\left(\xi_i, \eta_{j_{L_2-L_1}}^*\right)$ can be replaced by $\frac{p_{n+L_1}(\xi_i)\ p_{n+L_1}^*\left(\eta_{j_{L_2-L_1}}\right)}{h_{n+L_1}}$; furthemore, $K_{n+L_1+1}\left(\xi_i, \eta_{j_{L_2-L_1-1}}^*\right)$ can be replaced by $\frac{p_{n+L_1+1}(\xi_i)\ p_{n+L_1+1}^*\left(\eta_{j_{L_2-L_1-1}}\right)}{h_{n+L_1-1}}$ since the terms $p_{n+L_1}(\xi_i)$ are now proportional in the lines $n+L_1$ and $n+L_1+1$. The argument follows up to $N+L_2-1$; we may now write

$$
< \ n, L_2 > (\xi_i; \eta_i^*) = \prod_{i=0}^{L_2-1} \frac{h_{n+i}}{[n+i, L_1]^*(\eta_i)} \sum_{\{j_k\}=1}^{L_1} (-)^{L_1 L_2 - \sum j_k}
$$

$$
\prod_{k=1}^{L_2} [n+1+L_2-k, L_1-1]_{\widehat{j_k}}^*(\eta_i) \prod_{k=1}^{L_2-L_1} \frac{p_{n+L_2-k}^*\left(\eta_{j_k}\right)}{h_{n+L_2-k}}
$$

$$
\begin{vmatrix}
p_{n+L_2-1}(\xi_1) & ... & p_{n+L_2-1}\left(\xi_{L_2}\right) \\
... & ... & ... \\
p_{n+L_1}(\xi_1) & ... & p_{n+L_1}\left(\xi_{L_2}\right) \\
K_{n+L_1-1}\left(\xi_1, \eta_{j_{L_2-L_1+1}}^*\right) & ... & K_{n+L_1-1}\left(\xi_{L_2}, \eta_{j_{L_2-L_1+1}}^*\right) \\
... & ... & ... \\
K_{n+L_1-1}\left(\xi_1, \eta_{j_{L_2}}^*\right) & ... & K_{n+L_1-1}\left(\xi_{L_2}, \eta_{j_{L_2}}^*\right)
\end{vmatrix} \quad (A2)
$$

Concerning the indices $\{j_1, ..., j_{L_2-L_1}\}$ we develop $[n, L]^*(\eta_i)$ defined in (4) in regards to the elements $p_n^*(\eta_j)$

$$
[n, L]^*(\eta_i) = \sum_{j=1}^{L} (-)^{L-j}\ [n+1, L-1]_{\widehat{j}}^*(\eta_i)\ p_n^*(\eta_j) \qquad (A3)
$$

where the notation $[n+1, L-1]_{\widehat{j}}^*(\eta_i)$, already described below (61), means that the column relative to the variable η_j^* has been ommitted in the determinant; concerning the indices $\{j_{L_2-L_1+1}, ..., j_{L_2}\}$ we use the antisymmetry of the determinant to order the set $\left\{\eta_{j_{L_2-L_1+1}}^*, ..., \eta_{j_{L_2}}^*\right\}$ into the set $\{\eta_1^*, ..., \eta_{L_1}^*\}$. We get

$$
< \ n, L_2 > (\xi_i; \eta_i^*) = (-)^{\frac{L_1(L_1-1)}{2}} \prod_{i=0}^{L_1-1} \frac{h_{n+i}}{[n+i, L_1]^*(\eta_i)}
$$

$$
\times \begin{vmatrix}
\left[n+L_1, L_1-1\right]_{\widehat{1}}^*(\eta_i) & ... & \left[n+L_1, L_1-1\right]_{\widehat{L_1}}^*(\eta_i) \\
... & ... & ... \\
\left[n+1, L_1-1\right]_{\widehat{1}}^*(\eta_i) & ... & \left[n+1, L_1-1\right]_{\widehat{L_1}}^*(\eta_i)
\end{vmatrix}
$$

22

$$\times \begin{vmatrix} p_{n+L_2-1}(\xi_1) & ... & p_{n+L_2-1}(\xi_{L_2}) \\ ... & ... & ... \\ p_{n+L_1}(\xi_1) & ... & p_{n+L_1}(\xi_{L_2}) \\ K_{n+L_1-1}(\xi_1,\eta_1^*) & ... & K_{n+L_1-1}(\xi_{L_2},\eta_1^*) \\ ... & ... & ... \\ K_{n+L_1-1}(\xi_1,\eta_{L_1}^*) & ... & K_{n+L_1-1}(\xi_{L_2},\eta_{L_1}^*) \end{vmatrix} \qquad (A4)$$

It is the purpose of 3°) of part 1 of this appendix to prove that

$$\begin{vmatrix} \left[n+L_1,L_1-1\right]_{\widehat{1}}^*(\eta_i) & ... & \left[n+L_1,L_1-1\right]_{\widehat{L_1}}^*(\eta_i) \\ ... & ... & ... \\ \left[n+1,L_1-1\right]_{\widehat{1}}^*(\eta_i) & ... & \left[n+1,L_1-1\right]_{\widehat{L_1}}^*(\eta_i) \end{vmatrix}$$

$$= \quad (-)^{\frac{L_1(L_1-1)}{2}} \prod_{i=1}^{L_1-1} [n+i,L_1]^*(\eta_i) \qquad (A5)$$

Using the definition (25), we just finish the proof of (62).

2°) We consider $L_2 \leq L_1$. From (20) and (61) we write

$$< \quad n,L_2 > (\xi_i;\eta_i^*) = \prod_{i=0}^{L_2-1} \frac{h_{n+i}}{[n+i,L_1]^*(\eta_i)}$$

$$\times \sum_{\{j_k\}=1}^{L_1} (-)^{L_1L_2-\sum j_k} \prod_{k=1}^{L_2} [n+1+L_2-k,L_1-1]_{\widehat{j_k}}^*(\eta_i)$$

$$\times \begin{vmatrix} K_{n+L_2-1}(\xi_1,\eta_{j_1}^*) & ... & K_{n+L_2-1}(\xi_{L_2},\eta_{j_1}^*) \\ ... & ... & ... \\ K_{n+L_2-1}(\xi_1,\eta_{j_{L_2}}^*) & ... & K_{n+L_2-1}(\xi_{L_2},\eta_{j_{L_2}}^*) \end{vmatrix} \qquad (A6)$$

where we sum over all indices $\{j_1,...,j_{L_2}\}$ running from 1 to L_1. In (A6), the antisymmetry of the determinant in regards to the exchange of the variables $\eta_{j_i}^*$ is such that we may reconstruct a determinant from the sum of products $[n+1+L_2-k,L_1-1]_{\widehat{j_k}}^*(\eta_i)$

$$< \quad n,L_2 > (\xi_i;\eta_i^*) = \prod_{i=0}^{L_2-1} \frac{h_{n+i}}{[n+i,L_1]^*(\eta_i)} \sum_{1\leq j_1<...<j_{L_2}\leq L_1} (-)^{L_1L_2-\sum j_k}$$

$$\times \begin{vmatrix} [n+L_2,L_1-1]_{\widehat{j_1}}^*(\eta_i) & ... & [n+L_2,L_1-1]_{\widehat{j_{L_2}}}^*(\eta_i) \\ ... & ... & ... \\ [n+1,L_1-1]_{\widehat{j_1}}^*(\eta_i) & ... & [n+1,L_1-1]_{\widehat{j_{L_2}}}^*(\eta_i) \end{vmatrix}$$

$$\times \begin{vmatrix} K_{n+L_2-1}(\xi_1,\eta_{j_1}^*) & ... & K_{n+L_2-1}(\xi_{L_2},\eta_{j_1}^*) \\ ... & ... & ... \\ K_{n+L_2-1}(\xi_1,\eta_{j_{L_2}}^*) & ... & K_{n+L_2-1}(\xi_{L_2},\eta_{j_{L_2}}^*) \end{vmatrix} \qquad (A7)$$

In 3°) of part 1 of the appendix, we prove that for $\quad 1 \leq j_1 < ... < j_{L_2} \leq L_1$

$$
\begin{vmatrix}
[n+L_2, L_1-1]^*_{\widehat{j_1}} (\eta_i) & ... & [n+L_2, L_1-1]^*_{\widehat{j_{L_2}}} (\eta_i) \\
... & ... & ... \\
[n+1, L_1-1]^*_{\widehat{j_1}} (\eta_i) & ... & [n+1, L_1-1]^*_{\widehat{j_{L_2}}} (\eta_i)
\end{vmatrix}
$$

$$
= \quad (-)^{\frac{L_2(L_2-1)}{2}} \prod_{i=1}^{L_2-1} [n+i, L_1]^* (\eta_i) \quad [n+L_2, L_1-L_2]^*_{(\widehat{j_1},...,\widehat{j_{L_2}})} (\eta_i) \ (A8)
$$

where the notation $[n+L_2, L_1-L_2]^*_{(\widehat{j_1},...,\widehat{j_{L_2}})} (\eta_i)$ means that the columns relative to the variables $\eta^*_{j_1}, ..., \eta^*_{j_{L_2}}$ have been ommitted in the determinant. Consequently,

$$
< \quad n, L_2 > (\xi_i; \eta^*_i) = \frac{\prod_{i=0}^{L_2-1} h_{n+i}}{[n, L_1]^* (\eta_i)}
$$

$$
\times \sum_{1 \leq j_1 < ... < j_{L_2} \leq L_1} (-)^{\chi(L_1, L_2) - \sum j_k} \quad [n+L_2, L_1-L_2]^*_{(\widehat{j_1},...,\widehat{j_{L_2}})} (\eta_i)
$$

$$
\times \begin{vmatrix}
K_{n+L_2-1} \left(\xi_1, \eta^*_{j_1}\right) & ... & K_{n+L_2-1} \left(\xi_1, \eta^*_{j_{L_2}}\right) \\
... & ... & ... \\
K_{n+L_2-1} \left(\xi_{L_2}, \eta^*_{j_1}\right) & ... & K_{n+L_2-1} \left(\xi_{L_2}, \eta^*_{j_{L_2}}\right)
\end{vmatrix} \qquad (A9)
$$

where $\chi(L_1, L_2) = L_1 L_2 - \frac{L_2(L_2-1)}{2}$. From the definition (26), the above sum is nothing but $D^*_{n+L_2-1} (\eta_i; \xi^*_i)$, developed relatively to the bloc K_{n+L_2-1} by choosing the columns $\{j_1 < ... < j_{L_2}\}$ and summing over all possible permutations. This achieve the proof of (63).

3°) We prove the equation (A8) which is meaningful only if $L_1 \geq L_2$. We first note that if $L_1 = L_2$ equation (A8) is simply equation (A5) via the convention $[n, 0]_{(\widehat{1},...,\widehat{L_1})} = 1$.

We introduce $(L_1 - L_2)$ independant functions $N_i(z)$ for $i = 1, ..., L_1 - L_2$ and we consider the sum

$$
J = \sum_{\{\alpha_k\}=1}^{L_1} (-)^{L_1^2 - \sum \alpha_k} \quad p_{n+L_1-1} (\eta_{\alpha_1}) ... p_n \left(\eta_{\alpha_{L_1}}\right)
$$

$$
\times \begin{vmatrix}
N_{L_1-L_2} (\eta_{\alpha_1}) & ... & N_{L_1-L_2} \left(\eta_{\alpha_{L_1}}\right) \\
... & ... & ... \\
N_1 (\eta_{\alpha_1}) & ... & N_1 \left(\eta_{\alpha_{L_1}}\right) \\
[n+L_2, L_1-1]_{\widehat{\alpha_1}} (\eta_i) & ... & [n+L_2, L_1-1]_{\widehat{\alpha_{L_1}}} (\eta_i) \\
... & ... & ... \\
[n+1, L_1-1]_{\widehat{\alpha_1}} (\eta_i) & ... & [n+1, L_1-1]_{\widehat{\alpha_{L_1}}} (\eta_i)
\end{vmatrix} \qquad (A10)
$$

24

where we sum over all indices $\{\alpha_1, ..., \alpha_{L_1}\}$ running from 1 to L_1.

On one hand, a direct calculation shows that J is the determinant

$$
\begin{vmatrix}
X_{n+L_1-1,L_1-L_2} & ... & X_{n+L_2,L_1-L_2} & X_{n+L_2-1,L_1-L_2} & ... & X_{n,L_1-L_2} \\
... & ... & ... & ... & ... & ... \\
X_{n+L_1-1,1} & ... & X_{n+L_2,1} & X_{n+L_2-1,1} & ... & X_{n,1} \\
0 & ... & 0 & [n+L_2-1,L_1](\eta_i) & ... & ? \\
... & ... & ... & ... & ... & ... \\
0 & ... & 0 & 0 & ... & [n,L_1](\eta_i)
\end{vmatrix}
$$

$$(A11)$$

where

$$
X_{n,m} = \sum_{\alpha=1}^{L_1} (-)^{L_1-\alpha} \; p_n(\eta_\alpha) \; N_m(\eta_\alpha) \tag{A12}
$$

The question marks ? are easy to compute but useless and the zeroes come from the fact that we reconstruct determinants with two lines alike. The L_2 bottom lines of the determinant is a $[L_2 \times L_1 - L_2]$ rectangle of zeroes and a $[L_2 \times L_2]$ triangular matrix with zeroes below the diagonal and ? above. Consequently

$$
J = \prod_{i=0}^{L_2-1} [n+i,L_1](\eta_i) \begin{vmatrix} X_{n+L_1-1,L_1-L_2} & ... & X_{n+L_1-1,1} \\ ... & ... & ... \\ X_{n+L_2,L_1-L_2} & ... & X_{n+L_2,1} \end{vmatrix} \tag{A13}
$$

On the other hand, we may use the antisymmetry in the indices α_i of the determinant in (A10) to reconstruct a determinant with the $p's$ which is nothing but $[n,L_1](\eta_i)$; the quantity J is now obtained from a restricted sum $\sum_{1 \le \alpha_1 < ... < \alpha_{L_1} \le L_1}$ and such a restriction implies $\alpha_i = i$. We may write J as

$$
(-)^{\frac{L_1(L_1-1)}{2}} [n,L_1](\eta_i) \begin{vmatrix} N_{L_1-L_2}(\eta_1) & ... & N_{L_1-L_2}(\eta_{L_1}) \\ ... & ... & ... \\ N_1(\eta_1) & ... & N_1(\eta_{L_1}) \\ [n+L_2,L_1-1]_{\widehat{1}}(\eta_i) & ... & [n+L_2,L_1-1]_{\widehat{L_1}}(\eta_i) \\ ... & ... & ... \\ [n+1,L_1-1]_{\widehat{1}}(\eta_i) & ... & [n+1,L_1-1]_{\widehat{L_1}}(\eta_i) \end{vmatrix}
$$

$$(A14)$$

Then, given a set $\{j_k\}$ of L_2 indices such that $1 \le j_1 < ... < j_{L_2} \le L_1$, we define $\{s_p\}$ as the set of $(L_1 - L_2)$ indices complementary of $\{j_k\}$ in $\{1, ..., L_1\}$ and such that $1 \le s_1 < ... < s_{L_1-L_2} \le L_1$. Now, we look in both expressions (A13 and A14) of J for the coefficient of $N_{L_1-L_2}(\eta_{s_1})...N_1\left(\eta_{s_{L_1-L_2}}\right)$; in the expression (A14) we obtain

$$
(-)^{\frac{L_1(L_1-1)}{2}} (-)^{\frac{(L_1-L_2)(L_1-L_2+1)}{2} - \sum s_p} \quad [n,L_1](\eta_i)
$$

$$
\times \begin{vmatrix} [n+L_2,L_1-1]_{\widehat{j_1}}(\eta_i) & ... & [n+L_2,L_1-1]_{\widehat{j_{L_2}}}(\eta_i) \\ ... & ... & ... \\ [n+1,L_1-1]_{\widehat{j_1}}(\eta_i) & ... & [n+1,L_1-1]_{\widehat{j_{L_2}}}(\eta_i) \end{vmatrix} \tag{A15}
$$

In the expression (A13) the corresponding coefficient is found to be

$$
(-)^{L_1(L_1-L_2)-\sum s_p} \prod_{i=0}^{L_2-1} [n+i, L_1]\,(\eta_i)
$$

$$
\times
\begin{vmatrix}
p_{n+L_1-1}\left(\eta_{s_1}\right) & \cdots & p_{n+L_1-1}\left(\eta_{s_{L_1-L_2}}\right) \\
\cdots & \cdots & \cdots \\
p_{n+L_2}\left(\eta_{s_1}\right) & \cdots & p_{n+L_2}\left(\eta_{s_{L_1-L_2}}\right)
\end{vmatrix}
\tag{A16}
$$

If we compare (A15) and (A16) we obtain (A8) up to complex conjugaison.

Part 2: We calculate $D_n\left(\xi_i;\eta_i^*\right)$ and $D_n^*\left(\eta_i;\xi_i^*\right)$ defined in (25-26) in terms of the generalized Schur polynomials (33). From (25) and (14) we have in the case $L_1 \le L_2$

$$
D_n\left(\xi_i;\eta_i^*\right) = \sum_{\{\alpha_k\}=0}^{n} \prod_{i=1}^{L_1} \frac{p_{\alpha_i}^*\left(\eta_i\right)}{h_{\alpha_i}}
\begin{vmatrix}
p_{n+L_2-L_1}\left(\xi_1\right) & \cdots & p_{n+L_2-L_1}\left(\xi_{L_2}\right) \\
\cdots & \cdots & \cdots \\
p_{n+1}\left(\xi_1\right) & \cdots & p_{n+1}\left(\xi_{L_2}\right) \\
p_{\alpha_1}\left(\xi_1\right) & \cdots & p_{\alpha_1}\left(\xi_{L_2}\right) \\
\cdots & \cdots & \cdots \\
p_{\alpha_{L_1}}\left(\xi_1\right) & \cdots & p_{\alpha_{L_1}}\left(\xi_{L_2}\right)
\end{vmatrix}
\tag{A17}
$$

where we sum over L_1 indices α_k such that $0 \le \alpha_k \le n$. Because of the antisymmetry in the indices α_k we can reorganize the sum into a restricted sum and we introduce a second determinant

$$
D_n\left(\xi_i;\eta_i^*\right) = \sum_{0\le\alpha_{L_1}<\ldots<\alpha_1\le n} \prod_{i=1}^{L_1} \frac{1}{h_{\alpha_i}}
\begin{vmatrix}
p_{\alpha_1}^*\left(\eta_1\right) & \cdots & p_{\alpha_1}^*\left(\eta_{L_1}\right) \\
\cdots & \cdots & \cdots \\
p_{\alpha_{L_1}}^*\left(\eta_1\right) & \cdots & p_{\alpha_{L_1}}^*\left(\eta_{L_1}\right)
\end{vmatrix}
$$

$$
\times
\begin{vmatrix}
p_{n+L_2-L_1}\left(\xi_1\right) & \cdots & p_{n+L_2-L_1}\left(\xi_{L_2}\right) \\
\cdots & \cdots & \cdots \\
p_{n+1}\left(\xi_1\right) & \cdots & p_{n+1}\left(\xi_{L_2}\right) \\
p_{\alpha_1}\left(\xi_1\right) & \cdots & p_{\alpha_1}\left(\xi_{L_2}\right) \\
\cdots & \cdots & \cdots \\
p_{\alpha_{L_1}}\left(\xi_1\right) & \cdots & p_{\alpha_{L_1}}\left(\xi_{L_2}\right)
\end{vmatrix}
\tag{A18}
$$

Clearly, if $n < L_1-1$, there is not enough room for the indices α_k and $D_n\left(\xi_i;\eta_i^*\right)=0$; if $n=L_1-1$, the indices are fixed to $\alpha_k = L_1-k$ so that $D_{L_1-1}\left(\xi_i;\eta_i^*\right) = \prod_{k=0}^{L_1-1} \frac{1}{h_k}\,\Delta^*\left(\eta_i\right)\,\Delta\left(\xi_i\right)$ where Δ is the Vandermonde determinant (34). Now, for $n \ge L_1-1$ we introduce the generalized Schur polynomials defined in (33) and we write for $L_1 \le L_2$

$$
\frac{D_n\left(\xi_i;\eta_i^*\right)}{\Delta^*\left(\eta_i\right)\,\Delta\left(\xi_i\right)} = \sum_{\lambda\in[L_1\times(n-L_1+1)]} \prod_{k=1}^{L_1} \frac{1}{h_{\lambda_k+L_1-k}}
$$

26

$$\times P_\lambda^* (\eta_i) \; P_{[\{(L_2-L_1)\times(n-L_1+1)\}\cup\lambda]} (\xi_i) \qquad (A19)$$

where λ is a Young tableau in the rectangle $[L_1 \times (n - L_1 + 1)]$ with the rows of length $\lambda_k = \alpha_k - L_1 + k$.

A similar demonstration gives for $L_1 \geq L_2$ and $n \geq L_2 - 1$

$$\frac{D_n^* (\eta_i; {\xi_i}^*)}{\Delta^* (\eta_i) \; \Delta (\xi_i)} \;\; = \sum_{\lambda \in [L_2 \times (n-L_2+1)]} \prod_{k=1}^{L_2} \frac{1}{h_{\lambda_k+L_2-k}}$$
$$\times P_{[\{(L_1-L_2)\times(n-L_2+1)\}\cup\lambda]}^* (\eta_i) \;\; P_\lambda (\xi_i) \qquad (A20)$$

Again, $D_n^* (\eta_i; {\xi_i}^*) = 0$ if $n < L_2 - 1$ and $D_{L_2-1}^* (\eta_i; {\xi_i}^*) = \prod_{k=0}^{L_2-1} \frac{1}{h_k} \; \Delta^* (\eta_i) \; \Delta (\xi_i)$.

Part 3: We prove the relation (75). Given a Young tableau λ in the rectangle $[L \times N]$ and the corresponding Young tableau $\mu = \widetilde{\lambda}'$ (as defined in Sect. 3 (36)) in the rectangle $[N \times L]$ we establish a relation between the lengths of the rows of λ and the lengths of the rows of μ. The lengths of the rows of λ are defined by L numbers satisfying $N \geq \lambda_1 \geq ... \geq \lambda_L \geq 0$. We define the numbers ν_k as the numbers of rows of λ with length k. We have

$$\sum_{k=0}^{N} \nu_k \;\; = \;\; L \qquad (A21)$$

$$\sum_{k=0}^{N} k \, \nu_k \;\; = \;\; \sum_{i=1}^{L} \lambda_i = |\lambda| \qquad (A22)$$

where $|\lambda|$ is the surface of the Young tableau λ. Finally, we define

$$\sigma_k = \sum_{i=0}^{k} \nu_i \qquad (A23)$$

so that $\sigma_N = L$; the sequence $\{\sigma_k\}$ is a weakly increasing sequence as k runs from 0 to N. We have the following properties: for $0 \leq k \leq N$, if $\nu_k > 0$, then

$$\lambda_{L-\sigma_k+1} = \lambda_{L-\sigma_k+2} = ... = \lambda_{L-\sigma_{k-1}} \quad\;\; = k \qquad (A24)$$

with $\sigma_{-1} = 0$ by convention. Now, we define the numbers

$$\alpha_i = \lambda_i + L - i \quad\;\; \text{for } i = 1, ..., L \qquad (A25)$$

The sequence $\{\alpha_i\}$ is a strictly decreasing sequence as i runs from 1 to L. Because of (A24), If $\nu_k > 0$

$\alpha_{L-\sigma_k+1}$ to $\alpha_{L-\sigma_{k-1}}$ are consecutive numbers from $\sigma_k + k - 1$ to $\sigma_{k-1} + k$

$$(A26)$$

Clearly, the numbers α_i are all numbers from $N + L - 1$ to 0 at the exception of the set $\{\sigma_k + k\}$ for $k = 0, ..., N - 1$. We note that if $\nu_k = 0$ with $\nu_{k+1}\nu_{k-1} > 0$, we get two consecutive missing numbers $\sigma_k + k$ and $\sigma_{k-1} + k - 1 = \sigma_k + k - 1$; if p consecutive $\nu's$ are nul, we get $(p + 1)$ consecutive numbers missing in the set $\{\alpha_i\}$ and belonging to the set $\{\sigma_k + k\}$. The set $\{\sigma_k + k\}$ is a strictly increasing sequence as k runs from 0 to $N - 1$. We may write

$$\{\alpha_i\} = \{0, 1, 2, ..., N + L - 1\} - \left\{ \cup_{k=0}^{N-1} (\sigma_k + k) \right\} \qquad (A27)$$

We now consider the Young tableau λ' in the rectangle $[N \times L]$ and symmetric of λ with regards to the diagonal line. It is easy to convince oneself that

$$\lambda'_k = \sum_{i=k}^{N} \nu_i = L - \sigma_{k-1} \qquad \text{for } k = 1, ..., N \qquad (A28)$$

and consequently, if $\mu = \widetilde{\lambda'}$ the lengths of the rows of μ are

$$\mu_k = L - \lambda'_{N-k+1} = \sigma_{N-k} \qquad \text{for } k = 1, ..., N \qquad (A29)$$

The sequence $\{\mu_k\}$ is a weakly decreasing sequence as k runs from 1 to N. We define

$$\beta_k = \mu_k + N - k = \sigma_{N-k} + N - k \qquad \text{for } k = 1, ..., N \qquad (A30)$$

The sequence $\{\beta_k\}$ is a strictly decreasing sequence as k runs from 1 to N.

We just proved that

$$\begin{aligned}
\{\alpha_i\} \cup \{\beta_k\} &= \{0, 1, 2, ..., N + L - 1\} & (A31) \\
\{\alpha_i\} \cap \{\beta_k\} &= \phi & (A32)
\end{aligned}$$

and this concludes the proof of (75).

Part 4: This part of the appendix is devoted to the proof of the equations (93-95) and (97).

$1°$) We first mention a generalisation of Christoffel-Darboux relation. Given two sets of K variables $(\xi_1...\xi_K)$ and $(\eta_1...\eta_K)$, from (90-91) we may evaluate the following integrals as

$$\begin{aligned}
I_N &= \int ... \int \prod_{i=1}^{N} d\mu(x_i) \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^{N}\prod_{a=1}^{K} (x_i - \xi_a) \prod_{i=1}^{N}\prod_{b=1}^{K} (x_i - \eta_b) \\
&= N! \frac{\prod_{i=0}^{N-1} h_i}{\Delta(\xi_a, \eta_b)} [N, 2K](\xi_a, \eta_b) & (A33)
\end{aligned}$$

On the other hand, we may write

$$\prod_{i<j}(x_i - x_j) \prod_{i=1}^{N}\prod_{a=1}^{K}(x_i - \xi_a) = \frac{\Delta(x_i, \xi_a)}{\Delta(\xi_a)} \qquad (A34)$$

and similarly for the η_b part, we may introduce the Vandermonde determinants of the orthogonal polynomials $p_n(x)$ and integrate; we get

$$I_N = N! \frac{\prod_{i=0}^{N+K-1} h_i}{\Delta(\xi_a)\Delta(\eta_b)} \sum_{0 \le j_1 < ... < j_K \le N+K-1} \prod_{\alpha=1}^{K} \frac{1}{h_{j_\alpha}}$$

$$\times \begin{vmatrix} p_{j_K}(\xi_1) & ... & p_{j_K}(\xi_K) \\ ... & ... & ... \\ p_{j_1}(\xi_1) & ... & p_{j_1}(\xi_K) \end{vmatrix} \cdot \begin{vmatrix} p_{j_K}(\eta_1) & ... & p_{j_K}(\eta_K) \\ ... & ... & ... \\ p_{j_1}(\eta_1) & ... & p_{j_1}(\eta_K) \end{vmatrix} \qquad (A35)$$

The sum in (A35) is nothing but det $K_{N+K-1}(\xi_a, \eta_b)$ where

$$K_n(\xi_a, \eta_b) = \sum_{i=0}^{n} \frac{1}{h_i} p_i(\xi_a) \ p_i(\eta_b) \qquad (A36)$$

We just proved that

$$\frac{\Delta(\xi_a)\Delta(\eta_b)}{\Delta(\xi_a, \eta_b)} [N, 2K](\xi_a, \eta_b) = \prod_{i=N}^{N+K-1} h_i \quad \det K_{N+K-1}(\xi_a, \eta_b) \qquad (A37)$$

For $K = 1$, equation (A37) is nothing but Christoffel-Darboux relation

$$\frac{1}{\xi - \eta} \begin{vmatrix} p_{N+1}(\xi) & p_{N+1}(\eta) \\ p_N(\xi) & p_N(\eta) \end{vmatrix} = h_N \ K_N(\xi, \eta) \qquad (A38)$$

Also, for $N = 0$ we have

$$\Delta(\xi_a)\Delta(\eta_b) = \prod_{i=0}^{K-1} h_i \quad \det K_{K-1}(\xi_a, \eta_b) \qquad (A39)$$

2°) We wish to evaluate the integrals

$$R_n^{(b)}(\xi_a, y) = \int d\mu(x) \ \frac{\prod_{a=1}^{L}(x - \xi_a)}{x - y} \ K_n(\xi_b, x) \quad \text{for } b = 1, ..., L \qquad (A40)$$

Clearly,

$$f_0(y, \theta) = \int d\mu(x) \ \frac{1}{x - y} \ K_n(\theta, x) = 2i\pi \ H_n(\theta; y) \qquad (A41)$$

29

where $H_n(\theta, y)$ is defined in (94) in terms of the Cauchy transform. Also,

$$f_1(\xi, y, \theta) = \int d\mu(x) \; \frac{x - \xi}{x - y} \; K_n(\theta, x) = (y - \xi) \; 2i\pi \; H_n(\theta; y) + 1 \qquad (A42)$$

We observe the recurrence relation

$$f_L(\xi_1, ..., \xi_L, y, \theta) = (y - \xi_L) \; f_{L-1}(\xi_1, ..., \xi_{L-1}, y, \theta) + const(\xi_1, ..., \xi_{L-1}, \theta) \tag{A43}$$

where constant means y independent. It is easy to see that the constant in (A43) is nul if the two following conditions are fulfilled: first, $n \geq L - 1$, second $\theta = \xi_b$ for $b = 1, ..., L - 1$. Of course, $f_L(\xi_1, ..., \xi_L, y, \theta)$ is completely symmetric in the ξ's and the same result is valid for $\theta = \xi_L$ by simply performing the recurrence (A43) from $(y - \xi_{b \neq L})$. Consequently,

$$R_n^{(b)}(\xi_a, y) = \prod_{a=1}^{L} (y - \xi_a) \; \left[2i\pi \; H_n(\xi_b; y) + \frac{1}{y - \xi_b} \right] \qquad \text{for } b = 1, ..., L \leq n + 1$$

$$(A44)$$

Finally, using the same recurrence (A43) again, we may write for $b = 1, ..., L > n + 1$

$$\begin{aligned} R_n^{(b)}(\xi_a, y) &= \prod_{a=1}^{L} (y - \xi_a) \; \left[2i\pi \; H_n(\xi_b; y) + \frac{1}{y - \xi_b} \right] \\ &\quad + \Pi^{(b)}_{L-n-2}(y, \xi_a) \end{aligned} \qquad (A45)$$

where $\Pi^{(b)}_{L-n-2}(y, \xi_\alpha)$ is a polynomial in y of degree $L - n - 2$.

3°) Now, we prove (93) for $N > M$ and $L = M$. We introduce the Cauchy transform (83) and we write

$$[N - M, 2M](\xi_i; y_j) = \prod_{j=1}^{M} \left(\frac{1}{2i\pi} \int \frac{d\mu(x_j)}{x_j - y_j} \right) \; [N - M, 2M](\xi_i, x_j) \quad (A46)$$

From (A37), (A46) becomes

$$\begin{aligned} [N - M, 2M](\xi_i; y_j) &= (-)^M \prod_{k=N-M}^{N-1} h_k \prod_{j=1}^{M} \left(\frac{1}{2i\pi} \int \frac{d\mu(x_j)}{x_j - y_j} \prod_{i=1}^{M} (x_j - \xi_i) \right) \\ &\quad \times \det \; K_{N-1}(\xi_k, x_j) \end{aligned} \qquad (A47)$$

For each column of the determinant, we can perform the integration by applying (A40) and (A44)

$$\begin{aligned} [N - M, 2M](\xi_i; y_j) &= \left(\frac{-1}{2i\pi} \right)^M \prod_{k=N-M}^{N-1} h_k \prod_{j=1}^{M} \prod_{i=1}^{M} (y_j - \xi_i) \\ &\quad \times \det \; \left[2i\pi \; H_{N-1}(\xi_k; y_j) + \frac{1}{y_j - \xi_k} \right] \end{aligned} \qquad (A48)$$

30

This achieve the proof of equation (93).

Next, we prove (95) for $N \leq M$. The determinant $[N - M, L + M](\xi_i; y_j)$ can be developped in regards to the variables $p_q(y_j)$ that is in regards to the C_M^N corresponding minors. Each minor is characterized by a set of N missing variables $(y_{j_1}, ..., y_{j_N})$ and such a minor is multiplied in the original determinant by

$$[0, L + N](\xi_i; y_{j_\alpha}) = (-)^{LN} \ \Delta(\xi_i) \prod_{\alpha=1}^{N} \left(\frac{1}{2i\pi} \int \frac{d\mu(x_\alpha)}{x_\alpha - y_{j_\alpha}} \prod_{i=1}^{L} (x_\alpha - \xi_i) \right) \ \Delta(x_b)$$

$$(A49)$$

Then, in (A49) we transform the Vandermonde $\Delta(x_b)$ according to (A39)

$$[0, L + N](\xi_i; y_{j_\alpha})$$

$$= (-)^{LN} \frac{\Delta(\xi_i)}{\Delta(\theta_a)} \prod_{k=0}^{N-1} h_k \ \prod_{\alpha=1}^{N} \left(\frac{1}{2i\pi} \int \frac{d\mu(x_\alpha)}{x_\alpha - y_{j_\alpha}} \prod_{i=1}^{L} (x_\alpha - \xi_i) \right)$$

$$\times \det \ K_{N-1}(\theta_a, x_b) \qquad (A50)$$

where we have introduced N variables θ_a at wishes. However, if we choose the set $\{\theta_a\} \subset (\xi_1, ..., \xi_L)$ we may apply (A40) and (A45) in order to perform the integrations

$$[0, L + N](\xi_i; y_{j_\alpha}) = \frac{(-)^{LN}}{(2i\pi)^N} \frac{\Delta(\xi_i)}{\Delta(\theta_a)} \prod_{k=0}^{N-1} h_k \ \det \ U_{N-1}(\theta_a, y_{j_\alpha}) \qquad (A51)$$

where

$$U_{N-1}(\theta_a, y_{j_\alpha}) = \prod_{i=1}^{L} (y_{j_\alpha} - \xi_i) \ \left[2i\pi \ H_{N-1}(\theta_a; y_{j_\alpha}) + \frac{1}{y_{j_\alpha} - \theta_a} \right]$$

$$+ \Pi_{L-N-1}^{(a)}(y_{j_\alpha}, \xi_k) \qquad (A52)$$

and where the polynomial $\Pi_{L-N-1}^{(a)}(y_{j_\alpha}, \xi_k)$ is of degree $(L - N - 1)$ in y_{j_α} and is nul if $L < N - 1$. Now, if we replace $[0, L + N](\xi_i; y_{j_\alpha})$ by (A51) in the original determinant correspondingly to each minor we obtain

$$[N - M, L + M](\xi_i; y_j)$$

$$= \frac{(-)^{LN}}{(2i\pi)^N} \frac{\Delta(\xi_i)}{\Delta(\theta_a)} \prod_{k=0}^{N-1} h_k \begin{vmatrix} U_{N-1}(\theta_1, y_1) & ... & U_{N-1}(\theta_1, y_M) \\ ... & ... & ... \\ U_{N-1}(\theta_N, y_1) & ... & U_{N-1}(\theta_N, y_M) \\ p_{M-N-1}(y_1) & ... & p_{M-N-1}(y_M) \\ ... & ... & ... \\ p_0(y_1) & ... & p_0(y_M) \end{vmatrix} (A53)$$

If $M = N$ the part $p_q(y_j)$ is absent from the determinant (A53). We note that if $L \leq M$, the polynomial $\Pi^{(a)}_{L-N-1}(y_j, \xi_k)$ in $U_{N-1}(\theta_a, y_j)$ vanish in the determinant by linear combination of the $p_q(y_j)$ for $q = 0, ..., L - N - 1$. This achieve the proof of (95).

4°) In the case $N \geq M$, we calculate the expression J_N in (97) by successive derivations $\left(-\frac{\partial}{\partial \xi_i}\right)_{y_i = \xi_i}$ of the relation (93).

We must calculate

$$\left[\prod_{i=1}^{M}\left(-\frac{\partial}{\partial \xi_i}\right) A(y_b, \xi_a) \det\left\{(y_a - \xi_a)\left[2i\pi \; H_{N-1}(\xi_a; y_b) + \frac{1}{y_b - \xi_a}\right]\right\}\right]_{y_k = \xi_k} \tag{A54}$$

where

$$A(y_b, \xi_a) = \frac{\prod_{b=1}^{M}\prod_{a \neq b}(y_b - \xi_a)}{\Delta(\xi_a) \; \Delta(y_b)} \tag{A55}$$

In (A54), we distribute the derivatives on $A(y_b, \xi_a)$ and on the determinant; the derivative $\left(-\frac{\partial}{\partial \xi_i}\right)$ acts only on the line i of the determinant. After derivation, the condition $y_k = \xi_k$ transforms that line into

$$2i\pi H_{N-1}(\xi_i; \xi_j) + \frac{1}{\xi_j - \xi_i} \qquad \text{for } j \; \neq \; i \tag{A56}$$

$$2i\pi H_{N-1}(\xi_i; \xi_i) \quad \text{for } j \; = \; i \tag{A57}$$

Now, the lines of the determinant without derivatives are δ_{ij} when $y_k = \xi_k$. Consequently, (A54) may be written

$$\sum_I \left[\prod_{i \notin I}\left(-\frac{\partial}{\partial \xi_i}\right) A(y_b, \xi_a)\right]_{y_k = \xi_k} \det(I) \tag{A58}$$

where $I \subset (\xi_1, ..., \xi_M)$ and $\det(I)$ is a subdeterminant (A56-A57) with $i, j \in I$. Finally, for $J \subset (\xi_1, ..., \xi_M)$ we find

$$\left[\prod_{i \in J}\left(-\frac{\partial}{\partial \xi_i}\right) A(y_b, \xi_a)\right]_{y_k = \xi_k} = 0 \quad \text{if } card(J) \text{ is odd} \tag{A59}$$

and if $card(J)$ is even

$$\left[\prod_{i \in J}\left(-\frac{\partial}{\partial \xi_i}\right) A(y_b, \xi_a)\right]_{y_k = \xi_k} = (-)^{\frac{M(M-1)}{2}} \sum_{\text{all pairings in } J} \prod_{(i,j) \in J}\left(\frac{-1}{(\xi_j - \xi_i)^2}\right) \tag{A60}$$

32

The equations (A58-A59-A60) prove the result (97).

References

1. Dijkgraaf R. and Vafa C.: Nucl. Phys. **B644**, 3-39 (2002)
2. Verbaarschot J.J.M. and Wettig T.: Ann. Rev. Nucl. Part. Sci. **50,** 343-410 (2000)
3. Baik J., Deift P. and Strahov E.: J. Math. Phys. **44**, 3657-3670 (2003)
4. Akemann G. and Vernizzi G.: Nucl. Phys. **B660**, 532-556 (2003)
5. Berenstein D., Maldacena J. and Nastase H.: JHEP **0204**,013 (2002)
6. Eynard B. and Kristjansen C.: JHEP **0210**, 027 (2002)
7. Szegö G.: orthogonal polynomials, Am. Math. Soc., Colloquium publications **23**, Providence (1975)
8. Ginibre J.: J. Math. Phys. **6**, 440-449 (1965)
9. Brezin E. and Hikami S.: Commun. Math. Phys. **214**, 111-135 (2000)
10. Uvarov V. B.: USSR Comput. Math. Phys. **9**, n° 6, 25-36 (1969)
11. Fyodorov Y. and Strahov E.: J. of Phys. **A36**, 3203-3213 (2003)
12. Dyson F.J.: J.Math.Phys. **3**, 140-175 (1962)
13. Mehta M.L.: Random matrices, Academic Press, San Diego CA, USA (1991)